On viscosity solutions of certain Hamilton-Jacobi equations: Regularity results and generalized Sard's Theorems

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Abstract

Under usual assumptions on the Hamiltonian, we prove that any viscosity solution of the corresponding Hamilton-Jacobi equation on the manifold M is locally semiconcave and $C^{1,1}_{\mathrm{loc}}$ outside the closure of its singular set (which is nowhere dense in M). Moreover, we prove that, under additional assumptions and in low dimension, any viscosity solution of that Hamilton-Jacobi equation satisfies a generalized Sard theorem. In consequence, almost every level set of such a function is a locally Lipschitz hypersurface in M.

1 Introduction

Let M be a smooth manifold without boundary. We denote by TM (resp. T^*M) the tangent bundle of M, (x,v) a point in TM, and $\pi : TM \to M$ the canonical projection. Similarly, we denote by T^*M the cotangent bundle of M, (x,p) a point in T^*M, and $\pi^* : T^*M \to M$ the canonical projection. We will assume that the manifold M is equipped with a complete Riemannian metric g. For every $v \in T_xM$, we set $\|v\| := \sqrt{g_x(v,v)}$. And we denote by $\|\cdot\|$ the dual norm on T^*M. Let $H : T^*M \to \mathbb{R}$ be an Hamiltonian of class C^k (with $k \geq 2$) which satisfies the three following conditions:

(H1) (Uniform superlinearity) For every $K \geq 0$, there is $C^*(K) < \infty$ such that

$$\forall (x,p) \in T^*M, \quad H(x,p) \geq K\|p\| - C^*(K).$$

(H2) (Uniform boundedness in the fibers) For every $R \geq 0$, we have

$$A^*(R) := \sup \{H(x,p) \mid \|p\| \leq R\} < \infty.$$

(H3) (Strict Convexity in the fibers) For every $(x,p) \in T^*M$, the second derivative along the fibers $\frac{\partial^2 H}{\partial p^2}(x,p)$ is positive definite.

We recall that a continuous function $u : M \to \mathbb{R}$ is a *viscosity solution* of the Hamilton-Jacobi equation

$$H(x, d_xu) = 0, \quad \forall x \in M, \tag{1}$$

if the two following properties are satisfied:

(i) (u *viscosity subsolution* of (1)) For every $x \in M$, if $\phi : M \to \mathbb{R}$ is a C^1 function such that $\phi \geq u$ and $\phi(x) = u(x)$, then

$$H(x, d_x\phi) \leq 0.$$

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(ii) (*u viscosity supersolution* of (1)) For every $x \in M$, if $\psi : M \to \mathbb{R}$ is a C^1 function such that $\psi \le u$ and $\psi(x) = u(x)$, then

$$H(x, d_x\psi) \ge 0.$$

It is well-known that, under very general assumptions, any viscosity solution of a first or second-order partial differential equation is locally semiconcave on the state-space (see for instance [28]). Moreover, recent results by Li and Nirenberg (see [32]) show that, as soon as a viscosity solution of an Hamiltonan-Jacobi equation does satisfy a regular Dirichlet-type condition, then it is semiconcave and $C^{1,1}_{loc}$ outside a closed set with finite $\mathcal{H}^{n-1}$-measure. In addition, recent works by the author (see Appendix A) also show that, under appropriate assumptions, any viscosity solution of an Hamiltonan-Jacobi equation with Dirichlet conditions satisfies Sard-type theorems. The aim of the present paper is to show that, even in absence of boundary conditions, any viscosity solution of the stationary Hamilton-Jacobi equation (1) shares certain properties of regularity. The purpose of this paper is twofold. First, we prove regularity results for viscosity solutions of (1) and their singular sets. Then, we show that, under additional assumptions, the viscosity solutions of (1) satisfy generalized Sard's theorems.

Before stating our first result, we recall that, if $u : M \to \mathbb{R}$ is locally semiconcave on M (we refer the reader to the section 2.4.2 for the definition of the local semiconcavity), we call *singular set* of u, denoted by $\Sigma(u)$, the set of $x \in M$ where u is not differentiable. Our first result is the following:

Theorem 1. *Assume that* (H1), (H2) *and* (H3) *are satisfied, let* $u : M \to \mathbb{R}$ *be a viscosity solution of (1). Then the function* u *is locally semiconcave on* M*. Moreover, the singular set of* u *is nowhere dense in* M *and* u *is* $C^{1,1}_{loc}$ *on the open dense set* $M \setminus \overline{\Sigma(u)}$*.*

We mention that the semiconcavity and the $C^{1,1}_{loc}$ regularity outside $\overline{\Sigma(u)}$ are easy to obtain. The difficulty in proving the theorem above is to show that the set $\overline{\Sigma(u)}$ has empty interior. We notice that, in general, the Lebesgue measure of the closure of $\Sigma(u)$ has no reason to be zero. In [35], Mantegazza and Mennucci present the example of a compact convex set $S \in \mathbb{R}^2$ with a $C^{1,1}$ boundary for which the set $\overline{\Sigma(d_S)}$ (where d_S denotes the distance function to the set S in $\mathbb{R}^2$) has positive Lebesgue measure. This is well-known that d_S is a viscosity solution of the Hamilton-Jacobi equation $|d_x u(x)|^2 - 1 = 0 \quad \text{on} \quad \mathbb{R}^2 \setminus S$. Moreover, since S is convex with $C^{1,1}$ boundary, the signed distance function $\Delta_S : \mathbb{R}^2 \to \mathbb{R}$ defined as,

$$\forall x \in \mathbb{R}^2, \quad \Delta_S(x) = \begin{cases} d_S(x) & \text{if} \quad x \notin S \\ -d_{\mathbb{R}^2 \setminus S}(x) & \text{if} \quad x \in S, \end{cases}$$

is a viscosity solution of the eikonal equation

$$|d_x u(x)|^2 - 1 = 0 \quad \text{on} \quad \mathbb{R}^2.$$

Therefore, the counterexample of Mantegazza-Mennucci gives rise to an example of viscosity solution (1) whose the closure of the singular set has positive Lebesgue measure. However, we recall that, as soon as the viscosity solutions of (1) must satisfy a Dirichlet-type condition, we can obtain much more regularity results. In this spirit, by the classical method of characteristics and under additional assumptions on the data, several authors obtained results on the regularity of u and its singular set, see for instance [32], [35], [36], [45].

Let $u : M \to \mathbb{R}$ be a function which is locally Lipschitz on M, we call *critical point* of u, any $x \in M$ such that $0 \in \partial u(x)$ (here, $\partial u(x)$ denotes the Clarke generalized differential of u at x, see section 2.3.3) . We denote by $\mathcal{C}(u)$ the set of critical points of u in M and we say that u satisfies the generalized Sard Theorem if the set $u(\mathcal{C}(u))$ has Lebesgue measure zero in

$\mathbb{R}$. Since u is locally Lipschitz, the Clarke Implicit Function Theorem (see [11, Section 7.1]) implies that for every point x in M which is not critical, there exists a neighborhood $\mathcal{V}$ of x in M such that the level set $\{u(y) = u(x) \mid y \in \mathcal{V}\}$ is a locally Lipschitz hypersurface in M. Therefore, if u satisfies the generalized Sard Theorem, then almost every level set of u is a locally Lipschitz hypersurface in M. Generalized Sard's theorems have been recently used in [29], [39] and [40] to obtain regularity results on the level sets of distance functions in Riemannian and sub-Riemannian geometry. In the present paper, our aim is to show that in small dimension, sometimes under additional assumptions, any viscosity solution of (1) satisfies the generalized Sard Theorem. In fact, if the dimension of M equals 1 or 2, any locally semiconcave function on M satisfies the Sard Theorem (see Theorem 8). In dimension 3, we can prove the results below:

Theorem 2. *Let M be a real-analytic Riemannian manifold of dimension 3 and $H : T^*M \to \mathbb{R}$ be an Hamiltonian which is analytic on T^*M. Under the assumptions* (H1)-(H2)-(H3)*, if u is viscosity solution of (1), then the set $u(\mathcal{C}(u))$ has Lebesgue measure zero.*

Thanks to a phenomenon of propagation of critical points along the extremal, Theorem 2 can be extended naturally to the non-analytic case whenever the Hamiltonian has the form

$$H(x,p) = \frac{1}{2}\|p\|^2 - p(f(x)), \quad \forall (x,p) \in T^*M,$$

where f is a vector field of class C^4 on M. In fact, the following more general result holds.

Theorem 3. *Let M be a smooth manifold of dimension 3 and $H : T^*M \to \mathbb{R}$ be an Hamiltonian of class at least C^4 on T^*M satisfying* (H1)-(H3) *and one of the two following hypotheses:*

(H4) *For every $x \in M$, $H(x,0) = 0$.*

(H5) *For every $x \in M$, $H(x,0) = 0 \Longrightarrow \frac{\partial H}{\partial p}(x,0) = 0$ (in local coordinates).*

If u is viscosity solution of (1), then the set $u(\mathcal{C}(u))$ has Lebesgue measure zero.

In [21], Ferry presents the example of a closed subset $S \subset \mathbb{R}^4$ whose the distance function d_S does not satisfy the generalized Sard Theorem. Moreover, we know that d_S is a viscosity solution of the eikonal equation $|d_x u(x)|^2 - 1 = 0$ on the open set $\mathbb{R}^4 \setminus S$. Hence, in other terms, Ferry provides a counterexample to Theorem 2 in the case of a non-complete Riemannian manifold[1]. We provide in the last section of the present paper a true counterexample to Theorem 2 on the hyperbolic space of dimension 4. Furthermore, as for Theorem 1, we mention that as soon as a given viscosity solution of (1) must satisfy a Dirichlet-type condition, we can obtain, under additional assumptions on the data, generalized Sard's theorems. For example, we proved such a result for the case of the distance function to a set N in Riemannian geometry in [39] (compare [29]). In fact, this approach is easily extendable to many other situations. We provide in Appendix A a more general Sard's Theorem in the context of viscosity solutions of Hamilton-Jacobi equations with Dirichlet-type conditions.

Assumptions (H4) and (H5) in Theorem 3 are very restrictive. In fact, as the next result shows, Theorem 3 holds for generic Hamiltonians.

Theorem 4. *Let M be a smooth manifold of dimension 3 and $H_0 : T^*M \to \mathbb{R}$ be an Hamiltonian of class C^2 on T^*M satisfying* (H1)-(H3)*. Then, there is an open dense subset $\mathcal{O}$ of*

[1] As a matter of fact, the open set $\mathbb{R}^4 \setminus S$ is not complete with respect to the Euclidean metric in $\mathbb{R}^4$. In fact, we assumed that the Riemannian metric g is complete only for sake of simplicity. The reason being to avoid any blow-up phenomenon for the Euler-Lagrange flow. All the results presented in the present paper still hold if we drop the assumption of completeness. In particular, for every open set $\Omega \subset \mathbb{R}^n$ and every Hamiltonian $H : \Omega \times \mathbb{R}^n \to \mathbb{R}$ satisfying (H1)-(H3) on Ω, all our results apply for viscosity solutions of $H(x, d_x u) = 0$ on Ω.

$C^2(M,\mathbb{R})$ (in the Whitney C^2-topology), such that, for every $f \in \mathcal{O}$ and every $c \in \mathbb{R}$, any viscosity solution of

$$H_0(x, d_x u) + f(x) - c = 0, \quad \forall x \in M, \tag{2}$$

satisfies the generalized Sard Theorem.

The proofs of our generalized Sard theorems are strongly based on methods which were developed by Bates and Norton in [7] and [38]. Roughly speaking, all our Sard-type results use the fact that any locally semiconcave function on a surface satisfies the generalized Sard theorem, which is false in greater dimension. Moreover, it is not difficult to see that, under appropriate assumptions, the set to consider in order to estimate the size of the set of critical values of a given viscosity solution can be covered by a countable union of lipschitz hypersurfaces in M. This explains why, as soon as we work in dimension 3 we are able to obtain Sard-type results while this is impossible in greater dimension. We mention that these methods, especially the one of Norton, was used more extensively in [17] to prove specific results related to generalized Sard's theorems in the context of weak KAM theory.

We make clear that, if M is assumed to be compact, then for every Hamiltonian $H : T^*M \to \mathbb{R}$ satisfying (H1)-(H3), there is a unique value $c = c(H) \in \mathbb{R}$ (called the critical value or the Mañé critical value of H) for which the Hamilton-Jacobi equation

$$H(x, d_x u) - c = 0, \quad \forall x \in M, \tag{3}$$

admits viscosity solutions. Since the Hamiltonian $H - c$ also satisfies (H1)-(H3), all the results of this paper hold for viscosity solutions of (3) (also called weak KAM solutions). Since we adressed this problem from the weak KAM viewpoint, we restricted our attention to stationary Hamilton-Jacobi equations. It would be certainly interesting to develop the same kind of results in the context of parabolic first-order viscosity solutions and try to establish links with existing results on the regularity of propagating fronts such as those of Ley [31] and Barles, Ley [4].

The paper is organized as follows: In Section 2, we recall basic facts in calculus of variations, generalized differential calculus and semiconcavity theory. The proof of Theorem 1 occupies all Section 3. The proofs of Theorems 2, 3 and 4 are given in Section 4. Then we present in Section 5 a counterexample to Theorem 2 in dimension 4. Furthermore, we present in Appendix A, a general generalized Sard's Theorem for viscosity solutions of certain Hamilton-Jacobi equations with Dirichlet-type conditions.

Notations: Throughout this paper, we denote by $\langle \cdot, \cdot \rangle$ and $|\cdot|$, respectively, the Euclidean scalar product and norm in $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$ and any $r > 0$, we set $B(x,r) := \{y \in \mathbb{R}^n \mid |y-x| < r\}$ and $\bar{B}(x,r) := \{y \in \mathbb{R}^n \mid |y-x| \leq r\}$. We will also use the abbreviations $B_r := B(0,r)$, $\bar{B}_r := \bar{B}(0,r)$, $B := B_1$, and $\bar{B} := \bar{B}_1$. If A is a given subset of $\mathbb{R}^n$ and x is a point of $\mathbb{R}^n$, $d_A(x)$ will denote the distance from x to A. Thus $d_A : \mathbb{R}^n \to \mathbb{R}$ denotes the distance function to the set A. Finally, if A, B are two given subsets of $\mathbb{R}^n$, we will denote by $d_{\mathcal{H}}(A,B)$ the Hausdorff distance between A and B.

2 Preliminaries

2.1 Hamiltonian-Lagrangian duality

The Lagrangian $L : TM \to \mathbb{R}$ associated to H is defined by

$$\forall (x,v) \in TM, \quad L(x,v) := \max_{p \in T_x^* M} \{p(v) - H(x,p)\}.$$

Since H is C^2 and satisfies the three conditions (H1)-(H3), it is well-known that L is finite everywhere, of class C^2, and satisfies the following properties (we refer the reader to [18, Lemma 2.1] for the proofs):

(L1) (Uniform superlinerarity) For every $K \geq 0$, there is $C(K) < \infty$ such that

$$\forall (x,v) \in TM, \quad L(x,v) \geq K\|v\| - C(K).$$

(L2) (Uniform boundedness in the fibers) For every $R \geq 0$, we have

$$A(R) := \sup \{L(x,v) \mid \|v\| \leq R\} < \infty.$$

(L3) (Strict convexity in the fibers) For every $(x,v) \in TM$, the second derivative along the fibers $\frac{\partial^2 L}{\partial v^2}(x,v)$ is positive definite.

(L4) For every $R \geq 0$, we have

$$\sup \{\|p\| \mid (x,p) = \mathcal{L}(x,v), \|v\| \leq R\}, \sup \{\|v\| \mid (x,p) = \mathcal{L}(x,v), \|p\| \leq R\} < \infty.$$

In addition, we have the dual formula

$$\forall (x,p) \in T^*M, \quad H(x,p) = \max_{v \in T_xM} \{p(v) - L(x,v)\}.$$

The Legendre transform $\mathcal{L} : TM \to T^*M$ defined as,

$$\forall (x,v) \in TM, \quad \mathcal{L}(x,v) := \left(x, \frac{\partial L}{\partial v}(x,v)\right)$$

is a diffeomorphism of class C^1. Moreover we have

$$p(v) = H(x,p) + L(x,v) \Longleftrightarrow (x,p) = \mathcal{L}(x,v).$$

2.2 Calculus of variations and Euler-Lagrange flow

We recall briefly some basic facts in calculus of variations. We refer the reader to [16] for more details.

Let $x, y \in M$ and $T > 0$ be fixed. We denote by $\Omega_T(x)$ (resp. $\Omega_T(x,y)$) the set of locally Lipschitz curves $\gamma : [0,T] \to M$ such that $\gamma(0) = x$ (resp. $\gamma(0) = x$ and $\gamma(T) = y$). If $\gamma : [0,T] \to M$ is a locally Lipschitz curve, we define its *action* by

$$A_L(\gamma) := \int_0^T L(\gamma(s), \dot{\gamma}(s)) ds$$

We will say that a given curve $\gamma \in \Omega_T(x,y)$ minimizes the action if it satisfies the following property:

$$A_L(\gamma) \leq A_L(\gamma'), \quad \forall \gamma' \in \Omega_T(x,y).$$

Under assumptions (H1)-(H3), for every $x, y \in M$ and every $T > 0$, there is at least one curve $\gamma \in \Omega_T(x,y)$ which minimizes the action. In addition, this curve is necessarily a solution of the Euler-Lagrange equation (in local coordinates) :

$$\frac{d}{dt}\left[\frac{\partial L}{\partial v}(\gamma(s), \dot{\gamma}(s))\right] = \frac{\partial L}{\partial x}(\gamma(s), \dot{\gamma}(s)), \quad \forall s \in [a,b]. \tag{4}$$

In particular, the curve γ is C^2 on the interval $[0, T]$. The Euler-Lagrange equation generates a flow ϕ_t^L on TM which is C^1 and complete (see [18, Corollary 2.2]). This flow, called the *Euler-Lagrange flow* is defined by

$$\phi_t^L(x, v) := (\gamma_v(t), \dot{\gamma}_v(t)), \quad \forall t \in \mathbb{R},$$

where γ is the unique solution of (4) such that $\gamma(0) = x, \dot{\gamma}(0) = v$. In the sequel, we will call *extremal* of (4) on the interval $[a, b]$ (with $a < b \in \mathbb{R}$), any curve $\psi : [a, b] \to TM$ which satisfies

$$\psi(t) = \phi_t^L(\psi(0)), \quad \forall t \in [a, b].$$

Sometimes, we will as well call extremal any curve $\gamma : [a, b] \to M$ such that $(\gamma, \dot{\gamma}) : [a, b] \to TM$ is an extremal. Moreover, we will say that a given curve $\gamma : [a, b] \to M$ is a minimizing extremal if it minimizes the action between its end-points. Finally, we recall that the *energy* $E : TM \to \mathbb{R}$ associated with L is defined by

$$\forall (x, v) \in TM, \quad E(x, v) := (H \circ \mathcal{L})(x, v);$$

it is constant along the extremals.

2.3 Generalized sub- and superdifferentials

Here, we introduce several notions of generalized differentials on manifolds and general facts about them. We always refer the reader to [10], [12] or [41] for the proofs.

2.3.1 Viscosity sub- and superdifferentials

Let Ω be an open set in M and $u : \Omega \to \mathbb{R}$ be a continuous function. We call *viscosity subdifferential* of u at the point $x \in U$, the subset of T_x^*M defined by

$$D^-u(x) := \left\{ d\psi_x \mid \psi \in C^1(M) \text{ and } u - \psi \text{ attains a local minimum at } x \right\}.$$

Similarly, we call *viscosity superdifferential* of u at the point x, the subset of T_x^*M defined by

$$D^+u(x) := \left\{ d\phi_x \mid \phi \in C^1(M) \text{ and } u - \phi \text{ attains a local maximum at } x \right\}.$$

We notice that we can give a definition of viscosity sub- and supersolution of (1) in terms of sub- and superdifferentials. The continuous function $u : \Omega \to \mathbb{R}$ is a viscosity subsolution (resp. supersolution) of (1) on Ω if and only if for every $x \in \Omega$ and every $p \in D^+u(x)$ (resp. $p \in D^-u(x)$), we have

$$H(x, p) \leq 0 \quad (\text{resp. } H(x, p) \geq 0).$$

Let us recall some easy facts about the generalized differentials defined above.

Proposition 1. *For every $x \in \Omega$, the sets $D^-u(x)$ and $D^+u(x)$ are closed, convex and possibly empty.*

Proposition 2. *If u is differentiable at $x \in \Omega$, then $D^-u(x) = D^+u(x) = \{d_x u\}$.*

Proposition 3. *Let $x \in \Omega$, if both sets $D^-u(x)$ and $D^+u(x)$ are nonempty, then u is differentiable at x and $D^-u(x) = D^+u(x) = \{d_x u\}$.*

The viscosity subdifferential (resp. superdifferential) of u defines a multivalued mapping from Ω into the cotangent bundle T^*M. It is said to be locally bounded on Ω if for each $x \in \Omega$ there exists a neighborhood $\mathcal{V}$ of x such that $D^-u(\mathcal{V})$ is relatively compact in T^*M. The following result is standard.

Proposition 4. *The function u is locally Lipschitz on Ω if and only if the viscosity subdifferentials (resp. superdifferentials) of u are locally bounded on Ω.*

2.3.2 Limiting subdifferentials

Let Ω be an open set in M and $u : \Omega \to \mathbb{R}$ be a function which is locally Lipschitz on Ω. We call *limiting subdifferential* of u at the point $x \in \Omega$, the subset of T_x^*M defined by

$$\partial_L u(x) := \left\{ \lim p_k \mid x_k \to x, p_k \in D^- u(x_k) \right\}.$$

Since u is locally Lipschitz, by Proposition 4, we know that, for every $x \in M$, $\partial_L u(x)$ is a nonempty compact subset of T_x^*M. Moreover, by construction, the multivalued mapping $x \mapsto \partial_L u(x)$ is upper semicontinuous from Ω into T^*M.

2.3.3 Clarke's generalized differentials

Let Ω be an open set in M and $u : \Omega \to \mathbb{R}$ be a function which is locally Lipschitz on Ω. We call *Clarke's generalized differential*, or *generalized differential* for short, of u at the point $x \in \Omega$, the subset of T_x^*M defined by

$$\partial u(x) = \mathrm{co}\left(\partial_L u(x)\right).$$

(Here, $\mathrm{co}(A)$ denotes the convex hull of a subset A of T_xM.) By construction, the multivalued mapping $x \mapsto \partial u(x)$ is upper semicontinuous from Ω into T^*M. Moreover, we have for every $x \in \Omega$,

$$D^- u(x) \subset \partial_L u(x) \subset \partial u(x) \quad \text{and} \quad D^+ u(x) \subset \partial u(x).$$

Furthermore, if we define the limiting superdifferential of u at $x \in \Omega$ as

$$\partial^L u(x) := \left\{ \lim p_k \mid x_k \to x, p_k \in D^+ u(x_k) \right\},$$

then we have

$$\partial u(x) = \mathrm{co}\left(\partial^L u(x)\right).$$

The following theorem will be useful in the proof of generalized Sard's theorems (see [12, Theorem 2.4 p. 75]).

Theorem 5 (Lebourg's Mean Value Theorem)**.** *Let x, y belonging to $\mathbb{R}^n$, and suppose that u is locally Lipschitz on an open set containing the line segment $[x, y]$ in $\mathbb{R}^n$. Then there exists $t \in (0, 1)$ and $p \in \partial u\left(tx + (1-t)y\right)$ such that*

$$u(y) - u(x) = \langle p, y - x \rangle.$$

2.4 Locally semiconcave functions

We refer the reader to [10] and [41] for further details on semiconcavity.

2.4.1 Semiconcave functions in $\mathbb{R}^n$

Let Ω be an open and convex subset of $\mathbb{R}^n$, $u : \Omega \to \mathbb{R}$ be a continuous function, and C be a nonnegative constant. We say that u is C-*semiconcave* or *semiconcave* on Ω if

$$\mu u(y) + (1-\mu)u(x) - u(\mu x + (1-\mu)y) \leq \frac{\mu(1-\mu)C}{2}|x - y|^2, \tag{5}$$

for any $\mu \in [0, 1]$, and any $x, y \in \mathbb{R}^n$. The following result follows easily.

Proposition 5. *Under the assumptions above, the mapping $u : \Omega \to \mathbb{R}, x \mapsto u(x) - \frac{C}{2}|x|^2$ is concave on Ω.*

Therefore, if the function u is C-semiconcave on Ω, it can be written on Ω as the sum of a concave function and a smooth function:

$$\forall x \in \Omega, \quad u(x) = \left[u(x) - \frac{C}{2}|x|^2\right] + \frac{C}{2}|x|^2.$$

From this remark, we deduce that any semiconcave function is locally Lipschitz. The following result will be very useful to prove the semiconcavity of our viscosity solution.

Proposition 6. *Let Ω be an open and convex subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ be a function on Ω. If there is $\sigma \geq 0$ such that, for every $x \in \Omega$, there exists $p_x \in \mathbb{R}^n$ such that*

$$u(y) \leq u(x) + \langle p_x, y - x\rangle + \sigma|y - x|^2, \tag{6}$$

for any $y \in \Omega$, then u is (2σ)-semiconcave on Ω.

Proof of Proposition 6. Let $x, y \in \Omega$ and $\mu \in [0, 1]$ be fixed, set $\bar{x} := \mu x + (1 - \mu)y \in \Omega$. By assumption, there exists $p_{\bar{x}} \in \mathbb{R}^n$ such that

$$u(z) \leq u(\bar{x}) + \langle p_{\bar{x}}, z - \bar{x}\rangle + \sigma|z - \bar{x}|^2, \quad \forall z \in \Omega.$$

Applying that inequality with $z = x$ and $z = y$ and summing both quantities multiplied by μ and $(1 - \mu)$ respectively, yields

$$\begin{aligned} \mu u(x) + (1-\mu)u(y) &\leq u(\bar{x}) + \mu\sigma|x - \bar{x}|^2 + (1-\mu)\sigma|y - \bar{x}|^2 \\ &= u(\bar{x}) + \mu\sigma|(1-\mu)x - (1-\mu)y|^2 + (1-\mu)\sigma|\mu y - \mu x|^2 \\ &= u(\mu x + (1-\mu)y) + \frac{\mu(1-\mu)(2\sigma)}{2}|x - y|^2. \end{aligned}$$

We deduce that u is (2σ)-semiconcave on Ω. □

The converse result can be stated as follows; its proof is left to the reader.

Proposition 7. *Let Ω be an open and convex subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ be a function which is C-semiconcave on Ω. Then, for every $x \in \Omega$ and every $p \in D^+u(x)$, we have*

$$u(y) \leq u(x) + \langle p, y - x\rangle + \frac{C}{2}|y - x|^2, \quad \forall y \in \Omega, \tag{7}$$

In particular, $\partial D^+u(x) = \partial u(x)$, for every $x \in \Omega$.

The following result will be useful to obtain several characterization of the singular set of a given locally semiconcave function; we refer the reader to [41] for its proof.

Proposition 8. *Let Ω be an open and convex subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ be a function which is semiconcave on Ω. Then, for every $x \in \Omega$, u is differentiable at x if and only if $\partial u(x)$ is a singleton.*

Finally, we recall the following result which is fundamental to prove the $C^{1,1}_{loc}$ regularity in Theorem 1. We refer the reader to [10, Corollary 3.3.8] for its proof.

Proposition 9. *Let Ω be an open and convex subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ such that u and $-u$ are both C-semiconcave on Ω. Then u is of class $C^{1,1}$ on Ω and the map $x \mapsto \nabla u(x)$ is Lipschitz with constant C.*

2.4.2 Locally semiconcave functions on M

We first need to define the notion of *locally semiconcave* functions in $\mathbb{R}^n$. Let Ω be an open subset of $\mathbb{R}^n$; the function $u : \Omega \to \mathbb{R}$ is called locally semiconcave on Ω, if for every $x \in \Omega$, there is an open and convex neighbourhood of x where u is semiconcave. Now, if Ω is an open subset of M, the function $u : \Omega \to \mathbb{R}$ is said to be locally semiconcave on Ω, if for every $x \in \Omega$ there are an open neighbourhood $\mathcal{V}_x$ of x and a smooth diffeomorphism $\varphi_x : \mathcal{V}_x \to \varphi_x(\mathcal{V}_x) \subset \mathbb{R}^n$ such that $f \circ \varphi^{-1}$ is locally semiconcave on the open set $\widetilde{\mathcal{V}}_x = \varphi_x(\mathcal{V}_x) \subset \mathbb{R}^n$.

2.4.3 Singular set of a locally semiconcave function

Let $u : \Omega \to \mathbb{R}$ be a locally semiconcave function on an open set $\Omega \subset M$; we call the *singular set* of u, denoted by $\Sigma(u)$, the set of points in Ω where u is not differentiable, that is (from Proposition 8)

$$\Sigma(u) := \{x \in \Omega \mid \partial u(x) \text{ is not a singleton } \}.$$

For every $x \in \Omega$, the Clarke generalized differential of u at x is a nonempty compact convex subset of T_x^*M, then its dimension as a convex set is between 0 and n. This observation leads to a natural stratification of the singular set of u. We have

$$\Sigma(u) = \cup_{k=1}^{n} \Sigma^k(u),$$

where $\Sigma^k(u)$ is defined as

$$\Sigma^k(u) := \{x \in \Omega \mid \dim\left(\partial u(x)\right) = k\},$$

for each $k \in \{1, \cdots, n\}$. The following result is fundamental in the theory of locally semiconcave functions (see [10], [41], compare [2], [3]). Before stating the result, we recall that, given $r \in \{0, 1, \cdots, n\}$, the set $S \subset M$ is called a r-rectifiable set if there exists a locally Lipschitz function $f : \mathbb{R}^r \to M$ such that $S \subset f(\mathbb{R}^r)$. In addition, S is called countably r-rectifiable if it is the union of a countable family of r-rectifiable sets. The result is the following.

Theorem 6. *For every $k \in \{1, \cdots, n\}$, the set $\Sigma^k(u)$ is countably $(n-k)$-rectifiable.*

The following result on the propagation of singularities in any dimension is due to Albano and Cannarsa [1] (see also [10, Theorem 4.3.2 p. 89]). We stress that, in the statement of the result, $\partial\partial u(x)$ denotes the (topological) boundary of the set $\partial u(x)$ and $N_{D^+u(x)}(p)$ the normal cone to the convex set $D^+u(x)$ at p, that is, the set defined by

$$N_{D^+u(x)}(p) = \left\{p \in \mathbb{R}^n \mid \langle p, p - p' \rangle \geq 0, \forall p' \in D^+u(x)\right\}.$$

Moreover, $\mathcal{H}^\mu$ denotes the μ-dimensional Hausdorff measure (see [10]).

Theorem 7. *Let $u : \Omega \to \mathbb{R}$ be a locally semiconcave function on an open set $\Omega \subset \mathbb{R}^n$, $x \in \Sigma(u)$ be fixed, and $p \in \mathbb{R}^n$ be such that*

$$p \in \partial\partial u(x) \setminus \partial_L u(x);$$

define

$$\nu := \dim\left(N_{D^+u(x)}(p)\right).$$

Then a number $\rho > 0$ and a Lipschitz map $f : N_{\partial u(x)}(p) \cap B_\rho \to \Sigma(u)$ exist such that

$$f(q) = x - q + |q|h(q),$$

with $h(q) \to 0$ as $q \to 0$, and

$$\liminf_{r \to 0^+} r^{-\nu} \mathcal{H}^\nu \left(f\left(N_{\partial u(x)}(p) \cap B_\rho\right) \cap B_r(x)\right) > 0.$$

Moreover,

$$\operatorname{diam}\left(\partial u(f(q))\right) \geq \delta, \quad \forall q \in N_{\partial u(x)}(p) \cap B_\rho,$$

for some $\delta > 0$.

2.4.4 A generalized Sard theorem for locally semiconcave functions

Let $u : \Omega \to \mathbb{R}$ be a locally semiconcave function on an open set $\Omega \subset M$; we call *critical point* of u in Ω, any point $x \in M$ such that $0 \in \partial u(x)$. We denote by $\mathcal{C}(u)$ the set of critical points of u in Ω.

Theorem 8. *Let M be a smooth manifold of dimension ≤ 2 and $u : M \to \mathbb{R}$ be a locally semiconcave function. Then, the set $u(\mathcal{C}(u))$ has Lebesgue measure zero.*

This result does not hold in greater dimension. In fact, any function of class C^2 (hence locally semiconcave) on an open subet of $\mathbb{R}^n$ with $n \geq 3$ which is a counterexample to the classical Sard's theorem provides a counterexample (see for example [6] or [19]). The proof that we present here invokes an argument of Bates who proved in [7] that any function $g : \mathbb{R}^n \to \mathbb{R}$ of class $C^{n-1,1}$ satisfies Sard's Theorem (see also [21] where the same kind of argument already appeared).

Proof of Theorem 8. Let us first prove the result in the case where M has dimension 1. Without loss of generality, we can assume that we work in $\mathbb{R}$. From Proposition 7, by semiconcavity of u, there exists $\sigma \geq 0$ such that for every $x \in [0,1]$ and every $p \in D^+u(x) = \partial u(x)$,we have

$$u(y) \leq u(x) + \langle p, y - x\rangle + \sigma |y-x|^2, \quad \forall y \in [0,1].$$

This implies that for every pair x, y of critical points in $[0,1]$, we have

$$|u(y) - u(x)| \leq \sigma |y-x|^2. \tag{8}$$

Denote by A the set of critical points of u in the interval $[0,1]$. For every positive integer l, we can partition $[0,1]$ into l subintervals $I_1, \cdots, I_l$ of length $(1/l)$. From (8), for every $i \in \{1, \cdots, l\}$, the set $u(A \cap I_i)$ is included in an interval of length at most (σ/l^2). Hence we have

$$\mathrm{meas}(u(A)) \leq \sum_{i=1}^{l} \mathrm{meas}\left(u(A \cap I_i)\right) \leq \sum_{i=1}^{l} \frac{\sigma}{l^2} = \frac{\sigma}{l}.$$

Letting $l \to \infty$, we obtain that $\mathrm{meas}(u(A)) = 0$. We deduce easily the result for any locally semiconcave function on a manifold of dimension 1.

Let us now prove Theorem 8 in the case where M has dimension 2. We need the two following lemmas which will be useful in the proof of Theorem 2 as well.

Lemma 1. *Let $u : \mathbb{R}^n \to \mathbb{R}$ be a continuous function and D be a subset of $\mathbb{R}^n$ such that for every $\epsilon > 0$, there exists a covering of D by a countable union of balls B_i of diameter r_i such that $\sum_i r_i^2 < \epsilon$. If for every compact set $K \subset \mathbb{R}^n$, there exists $\sigma_K \geq 0$ such that*

$$|u(y) - u(x)| \leq \sigma_K |y-x|^2, \quad \forall x, y \in D \cap K, \tag{9}$$

then the set $u(D)$ has Lebesgue measure zero.

Proof of Lemma 1. It is sufficient to prove that for every compact set $K \subset \mathbb{R}^n$, the set $u(D \cap K)$ has Lebesgue measure zero. Let K be a compact subset of $\mathbb{R}^n$ and $\epsilon > 0$ be fixed. By assumption, there is a countable family of balls B_i of diameter r_i such that

$$D \cap K \subset \cup_i B_i \quad \text{and} \quad \sum_i r_i^2 < \epsilon.$$

By (9), for each i, the set $u(D \cap B_i)$ is included in an interval of length $(\sigma_K r_i^2)$. Thus we have

$$\mathrm{meas}(u(D \cap K)) \leq \sum_i \mathrm{meas}\left(u(D \cap B_i)\right) \leq \sum_i \sigma_K r_i^2 < \sigma_K \epsilon.$$

Letting ϵ tend to 0, we deduce that $u(D \cap K)$ has Lebesgue measure zero. □

Lemma 2. *Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function and E be a measurable subset of $\mathbb{R}^2$. If for every compact set $K \subset \mathbb{R}^2$, there is $\sigma_K \geq 0$ such that*

$$|u(y) - u(x)| \leq \sigma_K |y - x|^2, \quad \forall x, y \in E \cap K, \tag{10}$$

then the set $u(E)$ has Lebesgue measure zero.

Proof of Lemma 2. As before, it is sufficient to prove that for every compact set $K \subset \mathbb{R}^2$, the set $u(E \cap K)$ has Lebesgue measure zero. Hence, without loss of generality we can assume that E is included in a given compact set $K \subset \mathbb{R}^2$. From the Lebesgue density Theorem, E is the union of two measurable sets $E_1, E_2 \subset E$ such that E_1 has Lebesgue measure zero and any point of E_2 is a density point in E_2, that is

$$\forall x \in E_2, \quad \lim_{\delta \to 0} \frac{\mathrm{meas}(E_2 \cap Q(x,\delta))}{\mathrm{meas}(Q(x,\delta))} = 1,$$

where $Q(x,\delta)$ denotes a cube in $\mathbb{R}^n$ which contains x and with side length δ. This implies that for every positive integer P and any $x \in E_2$, there is a real number $\delta_P(x) > 0$ such that

$$\frac{\mathrm{meas}(E_2 \cap Q)}{\mathrm{meas}(Q)} \geq 1 - P^{-2}, \tag{11}$$

for any cube Q with center x and side length $|Q|$ less than $\delta_P(x)$. Moreover, it is clear[2] that for any y, z in the intersection of E_2 and such a cube Q, there is a sequence $x_0, \cdots, x_P$ of points in $E_2 \cap Q$ such that $x_0 = y, x_P = z$, and

$$|x_i - x_{i+1}| < \frac{2|Q|}{P}, \quad \forall i = 0, \cdots, P-1. \tag{12}$$

Thus, for every $x \in E_2$, every cube Q with center x and side length $|Q|$ less than $\delta_P(x)$, and every pair y, z in $E_2 \cap Q$, we have

$$\begin{aligned} |u(y) - u(z)| &\leq |u(x_0) - u(x_1)| + \cdots + |u(x_{P-1}) - u(x_P)| \\ &\leq \sigma_K |x_0 - x_1| + \cdots + \sigma_K |x_{P-1} - x_P| \\ &\leq \sigma_K P (2|Q|/P)^2 \\ &= \frac{4\sigma_K}{P} |Q|^2. \end{aligned}$$

This means that

$$\mathrm{meas}\left(u(E_2 \cap Q)\right) \leq \frac{4\sigma_K}{P} |Q|^2.$$

Since the set of cubes Q with centers at $x \in E_2$ and sides lengths $\delta < \delta_P(x)$ is a Vitali family for E_2 (see for example [19] or [37] for the notion of Vitali family), there is a countable subcollection $\{Q_i\}$ such that

$$\mathrm{meas}\left(E_2 \setminus \cup_i Q_i\right) = 0 \quad \text{and} \quad \sum_i \mathrm{meas}(Q_i) = \sum_i |Q_i|^2 < 2\mathrm{meas}(E_2).$$

[2] As a matter of fact, for every pair $y, z \in E_2 \cap Q$, the line segment $[y, z] \subset \mathbb{R}^2$ can be a covered by at most P subcubes of Q of diameter $|Q|/P$. By (11), each of these subcubes must contain a point in $E_2 \cap Q$. So, we can find $P+1$ points $x_0, \cdots, x_P \in E_2 \cap Q$ such that $x_0 = y, x_P = z$, and verifying (12).

From Lemma 1, we know that $\mathrm{meas}\left(u\left(E_2 \setminus \cup_i Q_i\right)\right) = 0$. Hence

$$\begin{aligned}
\mathrm{meas}(u(E_2)) &\leq \mathrm{meas}\left(u\left(E_2 \setminus \cup_i Q_i\right)\right) + \sum_i \mathrm{meas}\left(u\left(E_2 \cap Q_i\right)\right) \\
&\leq \sum_i \frac{4\sigma_K}{P} \mathrm{meas}(Q_i) \\
&\leq \frac{8\sigma_K \mathrm{meas}(E_2)}{P}.
\end{aligned}$$

Since P is arbitrary, $\mathrm{meas}(u(E_2))$ must vanish, and this completes the proof. □

Return to the proof of Theorem 8. Without loss of generality, we can assume that the function u is 2σ-semiconcave on a compact convex set $K \subset \mathbb{R}^2$. Denote by A the set of critical points of u in K. As before, we have

$$|u(y) - u(x)| \leq \sigma |y - x|^2, \quad \forall x, y \in A. \tag{13}$$

Hence we can apply Lemma 2 and then conclude. □

We note that the proof of Lemma 2 can be easily adapted to prove the following result.

Lemma 3. *Let $u : \mathbb{R}^n \to \mathbb{R}$ be a continuous function and E be a measurable subset of $\mathbb{R}^n$ with $n \geq 3$. If for every compact set $K \subset \mathbb{R}^n$, there is $C_K \geq 0$ such that*

$$|u(y) - u(x)| \leq C_K |y - x|^n, \quad \forall x, y \in E \cup K, \tag{14}$$

then the set $u(E)$ has Lebesgue measure zero.

3 Proof of Theorem 1

Throughout this section, $u : M \to \mathbb{R}$ is a continuous viscosity solution of (1), where the Hamiltonian H is assumed to be C^2 and to satisfy assumptions (H1)-(H3).

3.1 First properties of u

First, from the characterization of viscosity subsolutions in terms of viscosity superdifferentials, we know that for every $x \in M$ and every $p \in D^+u(x)$, we have $H(x,p) \leq 0$. By (H1) and Proposition 4, we easily deduce that u is locally Lipschitz on M. Moreover, since H is continuous on T^*M and convex in the p variable, we have

$$\forall x \in M, \quad \forall p \in \partial u(x), \quad H(x,p) \leq 0. \tag{15}$$

Since $\partial_L u(x) \subset \partial u(s)$ for every $x \in M$, this yields

$$\forall x \in M, \quad \forall p \in \partial_L u(x), \quad H(x,p) = 0. \tag{16}$$

The two following results will be useful; we refer the reader to the monograph [16] for their proofs.

Lemma 4. *Any locally Lipschitz curve $\gamma : [a,b] \to M$ satisfies the following property:*

$$u(\gamma(b)) - u(\gamma(a)) \leq \int_a^b L(\gamma(s), \dot{\gamma}(s)) ds. \tag{17}$$

Lemma 5. *For every $x \in M$, there exists a locally Lipschitz curve $\gamma_x : (-\infty, 0] \to M$ with $\gamma_x(0) = x$ such that*

$$u(x) - u(\gamma_x(t)) = \int_t^0 L(\gamma_x(s), \dot{\gamma}_x(s))ds, \quad \forall t \in]-\infty, 0]. \tag{18}$$

We notice that if a given locally Lipschitz curve $\gamma_x : (-\infty, 0] \to M$ satisfies (18), then for every $a < b \leq 0$, it minimizes the action between $\gamma_x(a)$ and $\gamma_x(b)$ on the interval $[a, b]$. Hence, it is an extremal.

3.2 Extremals and limiting subdifferentials

Lemma 6. *For every $x \in M$ such that $D^- u(x)$ is nonempty, there is a unique locally Lipschitz curve $\gamma_x : (-\infty, 0] \to M$ which satisfies $\gamma_x(0) = x$ and*

$$u(x) - u(\gamma_x(t)) = \int_t^0 L(\gamma_x(s), \dot{\gamma}_x(s))ds, \quad \forall t \in]-\infty, 0]. \tag{19}$$

Moreover, the set $D^- u(x)$ is a singleton and we have

$$\mathcal{L}\left(x, \dot{\gamma}_x(0)\right) = (x, p)\,, \tag{20}$$

where $D^- u(x) = \{p\}$.

Proof of Lemma 6. From Lemma 5, we know that there exists at least one curve $\gamma_x : (-\infty, 0] \to M$ with $\gamma_x(0) = x$ which satisfies (19). Furthermore, since $D^- u(x)$ is nonempty, there is a function $\psi : M \to \mathbb{R}$ of class C^1 such that

$$u(x) = \psi(x) \quad \text{ and } \psi(y) \leq u(y), \quad \forall y \in M.$$

For every $t \in (-\infty, 0)$ and every locally Lipschitz curve $\gamma : [t, 0] \to M$ such that $\gamma(t) = \gamma_x(t)$, we have (by (17) and (18)) :

$$\begin{aligned}
-\psi(\gamma(0)) + \int_t^0 L(\gamma(s), \dot{\gamma}(s))ds &\geq -u(\gamma(0)) + \int_t^0 L(\gamma(s), \dot{\gamma}(s))ds \\
&\geq -u(\gamma(t)) \\
&= -u(\gamma_x(t)) \\
&= -u(x) + \int_t^0 L(\gamma_x(s), \dot{\gamma}_x(s))ds \\
&= -\psi(\gamma_x(0)) + \int_t^0 L(\gamma_x(s), \dot{\gamma}_x(s))ds.
\end{aligned}$$

This means that for every fixed time $t \in (-\infty, 0)$, the curve $\gamma_x : [-t, 0] \to M$ minimizes the quantity

$$-\psi(\gamma(0)) + \int_t^0 L(\gamma(s), \dot{\gamma}(s))ds,$$

among all locally Lipschitz curves $\gamma : [t, 0] \to M$ such that $\gamma(t) = \gamma_x(t)$. By classical results in calculus of variations, we deduce that γ_x satisfies Euler-Lagrange equations and $\gamma_x(0) = d_x\psi$. We conclude easily by the Cauchy-Lipschitz Theorem. □

Lemma 7. *For every $x \in M$ and every $p \in \partial_L u(x)$, there is a unique locally Lipschitz curve $\gamma_{x,p} : (-\infty, 0] \to M$ which satisfies $\gamma_{x,p}(0) = x$, (19), and such that $\mathcal{L}\left(x, \dot{\gamma}_{x,p}(0)\right) = (x, p)$.*

Proof of Lemma 7. By definition of the limiting subdifferential, there is a sequence $(x_k, p_k) \in T^*M$ converging to (x, p) as k tends to ∞ such that $p_k \in D^-u(x_k)$ for every k. From Lemma 6, for every k, there is a minimizing extremal $\gamma_k : (-\infty, 0] \to M$ which satisfies

$$\gamma_k(0) = x_k, \tag{21}$$

$$u(x_k) - u(\gamma_k(t)) = \int_t^0 L(\gamma_k(s), \dot{\gamma}_k(s))ds, \quad \forall t \in]-\infty, 0], \tag{22}$$

and such that

$$\mathcal{L}\left(x_k, \dot{\gamma}_k(0)\right) = \left(x_k, p_k\right). \tag{23}$$

In particular, for every k and every $t \in (-\infty, 0]$, we have

$$\begin{aligned} E(\gamma_k(t), \dot{\gamma}_k(t)) &= E(\gamma_k(0), \dot{\gamma}_k(0)) \\ &= (H \circ \mathcal{L})\left(x_k, \dot{\gamma}_k(0)\right) \\ &= H(x_k, p_k) = 0. \end{aligned}$$

By (H1), there is $C^*(1) < \infty$ such that

$$\forall (x, p) \in T^*M, \quad H(x, p) \geq \|p\| - C^*(1).$$

By (22) together with (L4), we deduce that there is $D \geq 0$ such that, for every k and every $t \in (-\infty, 0]$,

$$\|\dot{\gamma}_k(t)\| \leq D.$$

By the Arzelá-Ascoli Theorem, the continuity of u and L, and (21)-(23), we deduce that there exists a locally Lipschitz curve $\gamma : (-\infty, 0] \to M$ with $\gamma(0) = x$ and such that (19) and (20) are satisfied. Such a curve is necessarily a solution of the Euler-Lagrange equation. The uniqueness is a direct consequence of the Cauchy-Lipschitz Theorem. $\square$

We note that since u is locally Lipschitz on M, its limiting subdifferentials are always nonempty. Therefore, for every $x \in M$, there exists a curve $\gamma_x :]-\infty, 0] \to M$ satisfying $\gamma_x(0) = x$, (19) and (20), where p belongs to $\partial_L u(x)$. Of course, this curve is an extremal.

3.3 Local semiconcavity of u

Lemma 8. *The function u is locally semiconcave on M.*

Proof of Lemma 8. Without loss of generality, we can assume that we work in $\mathbb{R}^n$. Let $\bar{x} \in \mathbb{R}^n$ be fixed and $\mathcal{V}$ be a compact neighborhood of $\bar{x}$ in $\mathbb{R}^n$. From the remark above, for every $x \in \mathcal{V}$, there is a curve $\gamma_x :]-\infty, 0] \to M$ satisfying $\gamma_x(0) = x$, and such that (19) and (20) hold for some $p \in \partial_L u(x)$. Moreover, since u is locally Lipschitz, the vectors $\dot{\gamma}_x(0)$ are uniformly bounded for $x \in \mathcal{V}$. Fix $t \in (-\infty, 0)$ and set $x_t := \gamma_x(t)$. For every $y \in \mathcal{V}$, define the smooth mapping $h_y : [t, 0] \to \mathbb{R}^n$ by

$$h_y(s) := \frac{y - x}{t}(t - s), \quad \forall s \in [t, 0],$$

and the smooth path $\tilde{\gamma}_y : [t, 0] \to M$ by

$$\tilde{\gamma}_y(s) := \gamma_x(s) + h_y(s), \quad \forall s \in [t, 0].$$

Note that $\tilde{\gamma}_y(0) = y$ and $\tilde{\gamma}_y(t) = x_t$. Therefore, by (17), we have

$$\begin{aligned} u(\tilde{\gamma}_y(0)) &\leq u(\tilde{\gamma}_y(t)) + \int_t^0 L(\tilde{\gamma}_y(s), \dot{\tilde{\gamma}}_y(s))ds \\ &\leq u(x_t) + \int_t^0 L(\gamma_x(s), \gamma_x(s))ds + \int_t^0 H_y(s)ds, \end{aligned}$$

where $H_y : [t, 0] \to \mathbb{R}$ is defined by

$$H_y(s) := L(\tilde{\gamma}_y(s), \dot{\tilde{\gamma}}_y(s)) - L(\gamma_x(s), \gamma_x(s))ds, \quad \forall s \in [t, 0].$$

Taking the last inequality together with (19) yields

$$u(y) \leq u(x) + \int_t^0 H_y(s)ds.$$

The function $\phi : \mathcal{V} \to \mathbb{R}$ defined as,

$$\phi(y) := \int_t^0 H_y(s)ds, \quad \forall y \in \mathcal{V},$$

is C^2 and satisfies $\phi(0) = 0$. We conclude by Proposition 6. □

As a corollary, we obtain the following theorem by Fathi (see [15]).

Corollary 1. *Under the assumptions of Theorem 1, if u is a C^1 solution of (1) then it is $C^{1,1}_{loc}$.*

Proof of Corollary 1. We already know that u is locally semiconcave on M. Furthermore, if we define the new Hamiltonian $H' : T^*M \to \mathbb{R}$ by

$$H'(x, p) := H(x, -p), \quad \forall (x, p) \in T^*M,$$

then, since u is C^1, the function $u' := -u$ satisfies

$$H'(x, d_x u') = 0, \quad x \in M.$$

Therefore since H' satisfies (H1)-(H3), we deduce that u' is locally semiconcave. Proposition 9 completes the proof. □

3.4 On the singular set of u

We recall that the singular set of u, denoted by $\Sigma(u)$, is defined as the set of points where u is not differentiable. Denote its closure by $\mathcal{S}$, that is,

$$\mathcal{S} := \overline{\Sigma(u)}.$$

The aim of this paragraph is to show that $\mathcal{S}$ has empty interior. First, we begin with preparatory results.

Lemma 9. *For every $x \in M$, every locally Lipschitz curve $\gamma_x : (-\infty, 0] \to M$ satisfying $\gamma_x(0) = x$ and (19), the two viscosity sub- and superdifferentials of u at the point $\gamma_x(t)$ coincide. In particular, u is differentiable at $\gamma_x(t)$.*

Proof of Lemma 9. Set $\bar{x} := \gamma_x(t)$. For every $y \in \mathcal{V}$, we define $h_y : [t,0] \to \mathbb{R}^n$ by

$$h_y(s) := s\left(\frac{y-\bar{x}}{t}\right), \quad \forall s \in [t,0],$$

and $\bar{\gamma}_y : [t,0] \to M$ by

$$\bar{\gamma}_y(s) := \gamma_x(s) + h_y(s), \quad \forall s \in [t,0].$$

By (17), we have

$$\begin{aligned} u(\bar{\gamma}_y(t)) &\geq u(\bar{\gamma}_y(0)) - \int_t^0 L(\bar{\gamma}_y(s), \dot{\bar{\gamma}}_y(s))ds \\ &\geq u(x) - \int_t^0 L(\gamma_x(s), \gamma_x(s))ds + \int_t^0 H_y(s)ds, \end{aligned}$$

where $H_y : [t,0] \to \mathbb{R}$ is defined by

$$H_y(s) := L(\gamma_x(s), \gamma_x(s))ds - L(\bar{\gamma}_y(s), \dot{\bar{\gamma}}_y(s)), \quad \forall s \in [t,0].$$

By (19) together with the inequality above, we obtain

$$u(y) \quad \geq \quad u(\bar{x}) + \int_t^0 H_y(s)ds.$$

The function $\psi : \mathcal{V} \to \mathbb{R}$ defined as,

$$\psi(y) := \int_t^0 H_y(s)ds, \quad \forall y \in \mathcal{V},$$

is of class C^2 and satisfies $\psi(0) = 0$. Proposition 6 completes the proof. □

As immediate corollaries, we have the following results:

Lemma 10. *For every $x \in M$ and every $\gamma_x : (-\infty, 0] \to M$ satisfying $\gamma_x(0) = x$ and (19), we have*

$$\forall t \in (-\infty, 0), \quad \gamma_x(t) \notin \Sigma(u). \tag{24}$$

In particular, for every $x \in \Sigma(u)$ and every $p \in \partial_L u(x)$, we have:[3]

$$\mathcal{L}(x,0) \neq (x,p).$$

It is not difficult to show that if $\gamma_x(-t)$ belongs to the set $\mathcal{S}$ for some $t > 0$, then $\gamma_x(-s) \in \mathcal{S}$ for all $s \leq t$. Moreover, we can prove that if for some $t > 0$ the function u is of class C^2 in a neighborhood of $\gamma_x(-t)$, then $\gamma_x(-s) \notin \mathcal{S}$ for any $s > 0$.

Lemma 11. *For every $x \in M$ and every $\gamma_x : (-\infty, 0] \to M$ satisfying $\gamma_x(0) = x$ and (19), there exists $p \in \partial_L u(x)$ such that*

$$\mathcal{L}\left(x, \dot{\gamma}_x(0)\right) = (x,p). \tag{25}$$

[3]In fact, this property is an easy consequence of the fact that, if $\mathcal{L}(x,0) = (x,p)$ then $\frac{\partial H}{\partial p}(x,p) = 0$. As a matter of fact, if p in $\partial_L u(x)$ is such that $\mathcal{L}(x,0) = (x,p)$, then it is the (unique) covector in T_x^*M where the convex mapping $p \mapsto H(x,p)$ attains its minimum. Hence, the set of $q \in T_x^*M$ where $H(x,q) = 0$ is reduced to the singleton $\{p\}$. In consequence, the limiting subdifferential of u at x is necessarily the singleton $\{p\}$, which proves that u is differentiable at x.

In consequence, at every point $x \in M$, there is a one-to-one mapping between the elements of $\partial_L u(x)$ and the curves $\gamma_x : (-\infty, 0] \to \mathbb{R}$ satisfying $\gamma_x(0) = x$ and (19). More precisely, the set of curves $\gamma_x : (-\infty, 0] \to \mathbb{R}$ starting at x and satisfying (19) corresponds exactly to the set of (projections of extremals which are) projections of extremals of the form $\psi : (-\infty, 0] \to TM$ with

$$\psi(t) = \phi_t^L \left(\mathcal{L}^{-1}(x,p)\right), \quad \forall t \leq 0,$$

where $p \in \partial_L u(x)$.

Proof of Lemma 11. Let $x \in M$ and $\gamma_x : (-\infty, 0] \to M$ satisfying $\gamma_x(0) = x$ and (19) be fixed. From Lemma 9, for every $t \in (-\infty, 0)$, the covector $p_t \in T^*_{\gamma_x(t)}M$ such that $\mathcal{L}(\gamma_x(t), \dot{\gamma}_x(t)) = x$ belongs to the viscosity subdifferential of u at $\gamma_x(t)$. By definition of the limiting subdifferential of u at x, this means that the covector $p \in T^*_x M$ such that $\mathcal{L}(x, \dot{\gamma}_x(0)) = (x,p)$ belongs to $\partial_L u(x)$. □

Proposition 10. *The set $\mathcal{S}$ has empty interior, that is, $\Sigma(u)$ is nowhere dense in M.*

The proof of Proposition 10 occupies the rest of the section.

Proof of Proposition 10. We argue by contradiction. So, we assume that $\mathcal{S}$ has not empty interior in M. Since

$$\overline{\Sigma(u)} = \cup_{k=1}^n \overline{\Sigma^k(u)},$$

either $\overline{\Sigma^1}$ has not empty interior, or there is $\bar{k} \in \{2, \cdots, n\}$ such that $\overline{\Sigma^{\bar{k}}}$ has not empty interior and such that all the sets $\overline{\Sigma^k(u)}$ have empty interior for $1 \leq k \leq \bar{k} - 1$. We set $\bar{k} = 1$ and

$$\mathcal{O} := \mathrm{Int}\left(\overline{\Sigma^1(u)}\right),$$

if $\overline{\Sigma^1}$ has not empty interior, and

$$\mathcal{O} := \mathrm{Int}\left(\overline{\Sigma^{\bar{k}}(u)}\right) \setminus \left(\cup_{k=1}^{\bar{k}-1} \overline{\Sigma^k(u)}\right)$$

otherwise (Here, $\mathrm{Int}(A)$ denotes the interior of the set $A \subset M$). In fact, without loss of generality, we can assume that everything happens in the Euclidean space $\mathbb{R}^n$.

Lemma 12. *For every $x \in \Sigma^{\bar{k}}(u) \cap \mathcal{O}$, we have*

$$\partial_L u(x) = \partial_{conv}\left(\partial u(x)\right)^4.$$

Proof of Lemma 12. Let $x \in \Sigma^{\bar{k}}(u) \cap \mathcal{O}$ be fixed. By the definitions of $\partial_L u(x)$ and $\partial u(x)$, we know that $\partial_L u(x) \subset \partial u(x)$. Moreover, since u is a viscosity solution of (1), we have that $H(x,p) = 0$ for every $p \in \partial_L u(x)$ (see (16)). Since $H(x,p) \leq 0$ for every $p \in \partial u(x)$ (see (15)) and H is strictly convex in the fibers (by (H3)), this means that

$$\partial_L u(x) \subset \partial_{\mathrm{conv}}\left(\partial u(x)\right). \tag{26}$$

If $\bar{k} = 1$, we have necessarily $\partial_L u(x) = \partial_{\mathrm{conv}}(\partial u(x))$ (because $\partial_L u(x)$ has two elements). So we can assume that $\bar{k} \geq 2$. If the inclusion (26) is strict, the set $\partial_{\mathrm{conv}}(\partial u(x)) \setminus \partial_L u(x)$ is not empty. But if p belongs to $\partial_{\mathrm{conv}}(\partial u(x)) \setminus \partial_L u(x)$, the dimension of the normal cone $N_{D^+u(x)}(p)$ is bigger than $n - \bar{k} + 1$. Thus, from Theorem 7, there is a Lipschitz manifold of dimension at least $n - \bar{k} + 1$ which propagates in $\Sigma(u)$ from x. But we know, by Theorem 6, that the sets $\Sigma^k(u)$ with $\bar{k} \leq k \leq n$ are countably $(n-k)$-rectifiable. Therefore x belongs necessarily to the

[4] Here, if A is a given convex subset of $\mathbb{R}^n$, $\partial_{\mathrm{conv}}(A)$ denotes the boundary of A in the affine subspace generated by A.

set $\cup_{k=1}^{\bar{k}-1}\overline{\Sigma^k(u)}$. This contradicts the fact that x belongs to $\Sigma^{\bar{k}}(u) \cap \mathcal{O}$. $\square$

Return to the proof of Proposition 10. Let $\bar{x} \in \Sigma^{\bar{k}} \cap \mathcal{O}$ and $\bar{p} \in \partial u(\bar{x})$ be such that

$$H(\bar{x}, \bar{p}) \leq H(\bar{x}, p), \quad \forall p \in \partial u(\bar{x}).$$

Notice that by strict convexity of H, such a vector is unique on $\partial u(\bar{x})$ and it does not belong to $\partial_{\mathrm{conv}}\left(\partial u(\bar{x})\right)$. Denote by H the linear subspace of dimension $\bar{k}$ such that $\partial u(\bar{x}) \subset \bar{p} + H$ and set $N := N_{D^+u(\bar{x})}(\bar{p})$. We note that since $\bar{p}$ does not belong to $\partial\mathrm{conv}\left(\partial u(\bar{x})\right)$, the set N is a linear subspace of $\mathbb{R}^n$ of dimension $n - \bar{k}$ which satisfies

$$N = H^{\perp}.$$

We set $\bar{N} := \bar{x} + N$. We define the map $\Psi : \partial u(\bar{x}) \to H$ by,

$$\Psi(p) := \pi_H\left(\frac{\partial H}{\partial p}(\bar{x}, p)\right), \quad \forall p \in \partial u(\bar{x}),$$

where π_H denotes the orthogonal projection on H.

Lemma 13. *Set $D := \Psi(\partial u(\bar{x}))$. The map Ψ is a homeomorphism from $\partial u(\bar{x})$ into D. In particular, D is homeomorphic to $\bar{D}_{\bar{k}}$ (the closed unit ball of dimension $\bar{k}$) and its boundary is homeomorphic to $\mathbb{S}^{\bar{k}-1}$ (the Euclidean sphere of dimension $\bar{k}-1$). Moreover, the vector 0 belong to the (relative) interior of D.*

Proof of Lemma 13. Since Ψ is the restriction of the mapping $p \in \mathbb{R}^n \mapsto \pi_H\left(\frac{\partial H}{\partial p}(\bar{x}, p)\right)$, it is sufficient to prove that Ψ is injective on $\partial u(\bar{x})$. We argue by contradiction. Let $p \neq p'$ in $\partial u(\bar{x})$ be such that $\Psi(p) = \Psi(p')$. This means that

$$\frac{\partial H}{\partial p}(\bar{x}, p) - \frac{\partial H}{\partial p}(\bar{x}, p') \in H^{\perp} = N,$$

which can be written as

$$\left\langle q, \frac{\partial H}{\partial p}(\bar{x}, p) - \frac{\partial H}{\partial p}(\bar{x}, p')\right\rangle = 0, \quad \forall q \in \partial u(\bar{x}).$$

Hence we can write,

$$\begin{aligned} H(\bar{x}, p) &= \left\langle p, \frac{\partial H}{\partial p}(\bar{x}, p)\right\rangle - L\left(\bar{x}, \frac{\partial H}{\partial p}(\bar{x}, p)\right) \\ &= \max_{v \in \mathbb{R}^n}\left\{\langle p, v\rangle - L(\bar{x}, v)\right\} \\ &\geq \left\langle p, \frac{\partial H}{\partial p}(\bar{x}, p')\right\rangle - L\left(\bar{x}, \frac{\partial H}{\partial p}(\bar{x}, p')\right). \end{aligned}$$

Which yields

$$L\left(\bar{x}, \frac{\partial H}{\partial p}(\bar{x}, p')\right) \geq L\left(\bar{x}, \frac{\partial H}{\partial p}(\bar{x}, p)\right).$$

By symmetry, we obtain the equality

$$L\left(\bar{x}, \frac{\partial H}{\partial p}(\bar{x}, p')\right) = L\left(\bar{x}, \frac{\partial H}{\partial p}(\bar{x}, p)\right),$$

which gives

$$\begin{aligned} H(\bar{x},p) &= \left\langle p, \frac{\partial H}{\partial p}(\bar{x},p)\right\rangle - L\left(\bar{x}, \frac{\partial H}{\partial p}(\bar{x},p)\right) \\ &= \left\langle p, \frac{\partial H}{\partial p}(\bar{x},p')\right\rangle - L\left(\bar{x}, \frac{\partial H}{\partial p}(\bar{x},p')\right). \end{aligned}$$

This contradicts the fact that the maximum in the formula

$$H(\bar{x},p) = \max_{v\in\mathbb{R}^n} \{\langle p,v\rangle - L(\bar{x},v)\}$$

is attained at a unique $v \in \mathbb{R}^n$. In consequence, we deduce that Ψ is one-to-one from $\partial u(\bar{x})$ into D. This means that D is homeomorphic to the compact convex set $\partial u(\bar{x})$ of dimension $\bar{k}$, which is itself homeomorphic to the closed unit ball of dimension $\bar{k}$. Furthermore, since $\bar{p}$ is the point p in $\partial u(\bar{x})$ where the mapping $p \mapsto H(\bar{x},p)$ attains its minimum and since we know that $\bar{p}$ does not belong to $\partial_{\mathrm{conv}}(\partial u(\bar{x}))$, the vector $\frac{\partial H}{\partial p}(\bar{x},\bar{p})$ is necessarily orthogonal to H. This means that $\Psi(\bar{p}) = 0$. In consequence, we have that 0 belong to the interior of D. The boundary of D being homeomorphic to the set $\partial_L u(\bar{x})$, it is of course homeomorphic to the sphere $\mathbb{S}^{\bar{k}-1}$. □

By the lemma above, we infer that there is $\bar{\delta} > 0$ such that D contains the $\bar{k}$-dimensional ball centered at the origin of radius $\bar{\delta}$. From now, we denote by $d_{\bar{N}}(\cdot)$ the distance function to the set $\bar{N}$. We notice that $d_{\bar{N}}$ is smooth on $\mathbb{R}^n \setminus \bar{N}$ with a gradient given by

$$\nabla d_{\bar{N}}(x) = \frac{x - \mathrm{Proj}_{\bar{N}}(x)}{|x - \mathrm{Proj}_{\bar{N}}(x)|}, \quad \forall x \in \mathbb{R}^n \setminus \bar{N}, \tag{27}$$

where $\mathrm{Proj}_{\bar{N}}$ denotes the projection on $\bar{N}$. In particular, for every $x \in \mathbb{R}^n \setminus \bar{N}$, the vector $\nabla d_{\bar{N}}(x)$ is orthogonal to N.

Lemma 14. *There is $\bar{\epsilon} > 0$ such that for every $p \in \partial_L u(\bar{x})$, the extremal $\gamma_{\bar{x},p} : (-\infty, 0] \to M$ starting at $\bar{x}$ and such that $\mathcal{L}(\bar{x}, \dot{\gamma}_{\bar{x},p}(0)) = (\bar{x},p)$ satisfies*

$$d_{\bar{N}}\left(\gamma_{\bar{x},p}(t)\right) \geq \frac{\bar{\delta}}{2}t, \quad \forall t \in [0,\bar{\epsilon}]. \tag{28}$$

Proof of Lemma 14. Fix $p \in \partial_L u(\bar{x})$, the extremal $\gamma_{\bar{x},p}$ defined in the statement of the lemma satisfies

$$\dot{\gamma}_{\bar{x},p}(0) = \frac{\partial H}{\partial p}(\bar{x},p) =: v$$

Moreover, the vector v can be decomposed as $\Psi(v) + (v - \Psi(v))$. We have

$$d_{\bar{N}}(\bar{x} + tv) = t|v|. \tag{29}$$

Now, we have

$$\begin{aligned} \frac{d}{dt} d_{\bar{N}}(\gamma_{\bar{x},p}(t)) &= \langle \nabla d_{\bar{N}}(\gamma_{\bar{x},p}(t)), \dot{\gamma}_{\bar{x},p}(t)\rangle \\ &= \langle \nabla d_{\bar{N}}(\gamma_{\bar{x},p}(t)), \Psi(\dot{\gamma}_{\bar{x},p}(t))\rangle + \langle \nabla d_{\bar{N}}(\gamma_{\bar{x},p}(t)), \dot{\gamma}_{\bar{x},p}(t) - \Psi(\dot{\gamma}_{\bar{x},p}(t))\rangle \\ &= \langle \nabla d_{\bar{N}}(\gamma_{\bar{x},p}(t)), \Psi(\dot{\gamma}_{\bar{x},p}(t))\rangle, \end{aligned}$$

since $\nabla d_{\bar{N}}(\gamma_{\bar{x},p}(t)) \in N^{\perp} = H$. The mapping $(t,p) \mapsto \Psi(\dot{\gamma}_{\bar{x},p}(t))$ is of class C^1. Hence we conclude easily by compactness of $\partial_L u(\bar{x})$, (29), and the fact that

$$\Psi(\dot{\gamma}_{\bar{x},p}(0)) = v.$$

$\square$

For every $t \in [0, \bar{\epsilon})$, we denote by $\mathcal{F}_{\bar{x}}(t)$ the front starting from $\bar{x}$ at time t, that is,

$$\mathcal{F}_{\bar{x}}(t) := \{\gamma_{\bar{x},p}(-t) \mid p \in \partial_L u(\bar{x})\}.$$

By the lemmas above, for every $t \in (0, \bar{\epsilon})$, the front $\mathcal{F}_{\bar{x}}(t)$ is homeomorphic to the sphere $\mathbb{S}^{\bar{k}-1}$ and satisfies

$$d_{\mathcal{H}}\left(\mathcal{F}_{\bar{x}}(t), \bar{N}\right) \geq \frac{\bar{\delta}}{2} t. \tag{30}$$

In particular, this implies easily that there is $\bar{\mu} > 0$ such that the front $\mathcal{F}_{\bar{x}}(\bar{\epsilon})$ does not intersect the cone

$$K_\mu = \{\bar{x} + q \mid |\pi_H(q)| \leq \mu |q|\}, \tag{31}$$

for every $\mu \in [0, \bar{\mu}]$. Moreover, we notice that, in view of the proof of the previous lemma, the front $\mathcal{F}_{\bar{x}}(\bar{\epsilon})$ is homotopic in $\mathbb{R}^n \setminus K_{\bar{\mu}/2}$ to a sphere $\mathcal{S}$ of dimension $\bar{k}-1$ whose the center belongs to the set $\bar{N}$.

From Theorem 7, there exist $\rho, \delta > 0$ and a Lipschitz map $f : N \cap B_\rho \rightarrow \Sigma(u)$ such that

$$f(q) = \bar{x} - q + |q| h(q), \tag{32}$$

where $h(q) \rightarrow 0$ as $q \rightarrow 0$ and

$$\operatorname{diam}\left(\partial u(f(q))\right) \geq \delta, \quad \forall q \in N \cap B_\rho. \tag{33}$$

In fact, taking ρ smaller if necessary, we can assume that $|h(q)| \leq \bar{\mu}/2$ for every $q \in N \cap B_\rho$. Let $\tilde{h} : N \rightarrow \mathbb{R}^n$ be a continuous function satisfying the following properties:

$$\tilde{h}(q) = 0, \quad \forall q \in N \setminus B_\rho, \tag{34}$$

$$\tilde{h}(q) = h(q), \quad \forall q \in B_{\frac{\rho}{2}}, \tag{35}$$

and

$$\left|\tilde{h}(q)\right| \leq \frac{\bar{\mu}}{2}, \quad \forall q \in N. \tag{36}$$

Define the new map $\tilde{f} : N \rightarrow \mathbb{R}^n$ by

$$\forall q \in N, \quad \tilde{f}(q) := \bar{x} + q + |q| \tilde{h}(q). \tag{37}$$

We notice that by construction, we have

$$\tilde{f}(q) = \bar{x} + q, \quad \forall q \in \mathbb{R}^n \setminus B_\rho, \tag{38}$$

and

$$\tilde{f}(q) \in \Sigma(u), \quad \forall q \in B_{\frac{\rho}{2}}. \tag{39}$$

Set

$$\tilde{\Sigma} := \tilde{f}(N).$$

We are now ready to complete the proof of Proposition 10. For sake of clarity, we prefer to distinguish two cases $\bar{k} = 1$ and $\bar{k} \in [2, n]$.

First Case: $\bar{k} = 1$
Recall that the point $\bar{x}$ belongs to the set $\Sigma^1(u) \cap \mathcal{O}$, where $\mathcal{O}$ was defined by

$$\mathcal{O} := \mathrm{Int}\left(\overline{\Sigma^1(u)}\right),$$

Our contradiction will come from the following result.

Lemma 15. *Let v be a nonzero vector in $\mathbb{R}^n$ and N be the hyperplan of equation*

$$\{q \in \mathbb{R}^n \mid \langle v, q\rangle = 0\}.$$

Let $F : N \to \mathbb{R}^n$ be a continuous mapping satisfying

$$F(q) = q, \quad \forall q \in N \setminus \bar{B}.$$

and $\bar{q} \in N$ with $|\bar{q}| > 1$ such that

$$[\bar{q} - v, \bar{q} + v] \cap F(N) = \{\bar{q}\}.$$

Then the two points $\bar{q} + v$ and $\bar{q} - v$ cannot be connected in $\mathbb{R}^n \setminus F(N)$.

The proof of this result follows an argument which was used by Feighn in [20] to prove that every proper C^2 immersion of codimension 1 separates $\mathbb{R}^n$ (see also [26]).

Proof of Lemma 15. We argue by contradiction. So, we assume that there is $\bar{q} \in N$ with $|\bar{q}| > 1$ and a continuous path $\Gamma : [0, 1] \to \mathbb{R}^n \setminus F(N)$ connecting $\bar{q} - v$ to $\bar{q} + v$, that is, satisfying $\Gamma(0) = \bar{q} - v$ and $\Gamma(1) = \bar{q} + v$. In fact, since $\Gamma([0, 1])$ is a compact subset of the open set $\mathbb{R}^n \setminus F(N)$, smoothing Γ if necessary, we may assume that the path Γ is C^∞ and satisfies

$$d_{F(N)}(\Gamma(t)) > \epsilon, \quad \forall t \in [0, 1], \tag{40}$$

where ϵ is some positive constant. From the C^0-dense h-principle for immersions (see [44, Theorem 4.2 p. 52] or [23]), there is an C^∞ immersion $\tilde{F} : N \to \mathbb{R}^n$ which satisfies

$$|\tilde{F}(q) - F(q)| \leq \epsilon, \quad \forall q \in N, \tag{41}$$

and

$$\tilde{F}(q) = F(q), \quad \forall q \in N \setminus \bar{B}. \tag{42}$$

From (40) and (41), we know that Γ is a smooth path connecting $\bar{q} - v$ to $\bar{q} + v$ in the complement of $\tilde{F}$ in $\mathbb{R}^n$. Define $\tilde{\Gamma} : [0, 1] \to \mathbb{R}^n$ by

$$\forall t \in [0, 1], \quad \tilde{\Gamma}(t) := \begin{cases} \Gamma(2t) & \text{if} \quad 0 \leq t \leq 1/2, \\ \bar{q} + (3 - 4t)v & \text{if} \quad 1/2 \leq t \leq 1. \end{cases}$$

Notice that $\tilde{\Gamma}(0) = \bar{q} - v, \tilde{\Gamma}(1/2) = \bar{q} + v$ and $\tilde{\Gamma}(1) = \bar{q} - v$. Therefore, moving (in case $\tilde{\Gamma}$ intersects Γ) and smoothing the path $\tilde{\Gamma}$ if necessary, this construction permits to extend Γ to an embedding $\tilde{\Gamma} : \mathbb{S}^1 \to \mathbb{R}^n$ such that

$$\tilde{\Gamma}\left(\mathbb{S}^1\right) \cap \tilde{F}(N) = \{\bar{q}\}.$$

The embedded circle $\tilde{\Gamma}(\mathbb{S}^1)$ is homotopic to a point in $\mathbb{R}^n$. Hence, by smoothing the homotopy, $\tilde{\Gamma}$ is the restriction to the boundary of a map Λ of the 2-disc $\bar{D}_2$ into $\mathbb{R}^n$. Without loss of generality, we can assume that Λ is transverse to the immersed manifold $\tilde{F}(N)$ (see [22]). Set

$$P(\tilde{F},\Lambda) := \left\{(q,\lambda) \in N \times \bar{D}_2 \mid \tilde{F}(q) = \Lambda(\lambda)\right\}.$$

Since Λ is transverse to the immersed manifold $\tilde{F}(N)$, the set $P(\tilde{F},\Lambda)$ is indeed an immersed smooth compact manifold of dimension 1 with a boundary given by

$$\partial P(\tilde{F},\Lambda) = P(\tilde{F},\tilde{\Gamma}) = \{\bar{q}\}.$$

But such a manifold is necessarily a finite disjoint union of circles and segments, so that its boundary must consist of an even number of points. This contradicts the fact that $\partial P(\tilde{F},\Lambda) = \{\bar{q}\}$ and then completes the proof of the lemma. □

Return to the proof of Proposition 10. From (36)-(37), the set $\tilde{\Sigma} = \tilde{f}(N)$ is included in the cone $K_{\bar{\mu}/2}$ defined by (31). Moreover, since $\bar{x}$ belongs to $\mathcal{O}$, the interior of $\overline{\Sigma^1(u)}$, there is a sequence of points $\{x_l\}$ in $\Sigma^1(u)$ which converges to $\bar{x}$ and which is included in the cone

$$\{\bar{x} + q \mid |\pi_H(q)| > \bar{\mu}|q|\}.$$

For every l and every $t > 0$, the front starting from x_l at time t defined as,

$$\mathcal{F}_l(t) := \{\gamma_{\bar{x}_l,p}(-t) \mid p \in \partial_L u(\bar{x}_l)\},$$

is simply a pair of points. Moreover, by regularity of the Euler-Lagrange flow together with the upper semicontinuity of the limiting subdifferential of u, if l is large enough, $\mathcal{F}_l(\bar{\epsilon})$ does not intersect the cone $K_{\bar{\mu}/2}$ (because it must be close to $\mathcal{F}_{\bar{x}}(\bar{\epsilon})$ which does not intersect K_μ for $\mu \in [0,\bar{\mu}]$). But, we know by Lemma 10, that the front $\mathcal{F}_l(t)$ never meets $\Sigma(u)$ in positive times. Taking $\bar{\epsilon}$ smaller if necessary, this also means that the fronts $\mathcal{F}_l(t)$ do not intersect the set $\tilde{\Sigma}$ for $t \in [0,\bar{\epsilon}]$. Since the two fronts $\mathcal{F}_{\bar{x}}(\bar{\epsilon})$ and $\mathcal{F}_l(\bar{\epsilon})$ are close (in term of Hausdorff distance), they are homotopic in $\mathbb{R}^n \setminus K_{\bar{\mu}/2}$. In addition, from one hand, we know that the front $\mathcal{F}_{\bar{x}}(\bar{\epsilon})$ is homotopic to $\mathcal{S}$ in $\mathbb{R}^n \setminus K_{\bar{\mu}/2}$. On the other hand, we know that, for l large enough, the front $\mathcal{F}_l(\bar{\epsilon})$ is homotopic to the point x_l in $\mathbb{R}^n \setminus \tilde{\Sigma}$. In conclusion, for l large enough, we can construct a homotopy which contracts the sphere $\mathcal{S}$ to the point x_l in $\mathbb{R}^n \setminus \tilde{\Sigma}$. This contradicts Lemma 15.

Second Case: $\bar{k} > 1$
Recall that the point $\bar{x}$ belongs to the set $\Sigma^{\bar{k}}(u) \cap \mathcal{O}$, where $\mathcal{O}$ was defined by

$$\mathcal{O} = \mathrm{Int}\left(\overline{\Sigma^{\bar{k}}(u)}\right) \setminus \left(\cup_{k=1}^{\bar{k}-1}\overline{\Sigma^k(u)}\right)$$

Our contradiction will come from the following result.

Lemma 16. *Let N an linear space of codimension $\bar{k}$ in $\mathbb{R}^n$ and $F : N \to \mathbb{R}^n$ be a continuous mapping satisfying*

$$F(q) = q, \quad \forall q \in N \setminus \bar{B}.$$

Let $\mathcal{D}$ be a disc of dimension $\bar{k}$ centered at $\bar{q} \in N$, included in $\mathbb{R}^n \setminus \bar{B}$ and such that

$$\mathcal{D} \cap F(N) = \{\bar{q}\}.$$

Then its boundary $\mathcal{S}$ is not homotopic to a point in $\mathbb{R}^n \setminus F(N)$.

Our proof is taken from [25] where Hirsch extends from codimension 1 to codimension k the result by Feighn (see also [26]).

Proof of Lemma 16. We argue by contradiction. So, we assume that the $\bar{k}-1$-sphere $\mathcal{S}$ is homotopic to a point in $\mathbb{R}^n \setminus F(N)$. In fact, by smoothing the homotopy, we can see $\mathcal{S}$ as the boundary of a $\bar{k}$-disc $\mathcal{D}'$ which does not intersect $F(N)$. Since $\mathcal{D}'$ is a compact subset of $\mathbb{R}^n \setminus F(N)$, there exists $\epsilon > 0$ such that

$$d_{\mathcal{H}}(\mathcal{D}', F(N)) > \epsilon. \tag{43}$$

From the C^0-dense h-principle for immersions (see [44, Theorem 4.2 p. 52] or [23]), there is an C^∞ immersion $\tilde{F}: N \to \mathbb{R}^n$ which satisfies

$$|\tilde{F}(q) - F(q)| \leq \epsilon, \quad \forall q \in N, \tag{44}$$

and

$$\tilde{F}(q) = F(q), \quad \forall q \in N \setminus \bar{B}. \tag{45}$$

From (43) and (44), we know that $\mathcal{D}'$ does not intersect the immersed manifold $\tilde{F}(N)$. Smoothing the set $\mathcal{D} \cup \mathcal{D}'$, we obtain a smooth map $\psi : \mathbb{S}^{\bar{k}} \to \mathcal{D} \cup \mathcal{D}'$ which sends the northern hemisphere to $\mathcal{D}$ and the southern hemisphere to $\mathcal{D}'$. This mapping can be extended into a smooth map $\Psi : B^{\bar{k}+1} \to \mathbb{R}^n$ which is generically transverse to the embedded manifold $\tilde{F}(N)$. Set

$$P(\tilde{F}, \Psi) := \left\{ (q, z) \in N \times B^{\bar{k}+1} \mid \tilde{F}(q) = \Psi(z) \right\}.$$

Since Ψ is transverse to $\tilde{F}(N)$, $P(\tilde{F}, \Psi)$ is indeed an immersed smooth compact manifold of dimension 1 with a boundary given by

$$\partial P(\tilde{F}, \Psi) = P\left(\tilde{F}, \psi\left(\mathbb{S}^{\bar{k}}\right)\right) = \{\bar{q}\}.$$

We conclude as in the proof of Lemma 15. □

Return to the proof of Proposition 10. As before, since $\bar{x}$ belongs to $\mathcal{O}$, there is a sequence of points $\{x_l\}$ in $\Sigma^{\bar{k}}(u)$ which converges to $\bar{x}$ and which is included in the cone

$$\{\bar{x} + q \mid |\pi_H(q)| > \bar{\mu}|q|\}.$$

For every l and every $t > 0$, the front starting from x_l at time t defined as,

$$\mathcal{F}_l(t) := \{\gamma_{\bar{x}_l, p}(-t) \mid p \in \partial_L u(\bar{x}_l)\},$$

is homeomorphic to the sphere $\mathbb{S}^{\bar{k}-1}$. By regularity of the Euler-Lagrange flow together with the upper semicontinuity of the limiting subdifferential of u, if l is large enough, the front $\mathcal{F}_l(t)$ does not intersect the set $\tilde{\Sigma}$ for $t \in [0, \bar{\epsilon}]$ (with $\bar{\epsilon}$ small enough). We deduce as in the first case that, for l large enough, the sphere $\mathcal{S}$ is homotopic to the point x_l in $\mathbb{R}^n \setminus \tilde{\Sigma}$. This contradicts Lemma 16 and then completes the proof of Proposition 10. □

4 Proofs of generalized Sard's theorems

4.1 A preparatory lemma

We know from Proposition 8 that Sard's Theorem always (under assumptions (H1)-(H3)) holds in dimension two. Hence, in the proofs of Theorems 2, 3 and 4, we can assume that M has dimension three. Moreover, since M can be covered by a countable union of local charts and since u is locally semiconcave on M, we can assume that we work in a (relatively compact) open subset Ω of $\mathbb{R}^3$, and that there is $\sigma > 0$ such that we have for every pair $x, y \in \mathcal{C}(u)$,

$$|u(y) - u(x)| \leq \sigma |y - x|^2. \tag{46}$$

We also assume from now that the Hamiltonian H is at least C^4. We will often use the following result.

Lemma 17. *Let $\mathcal{O} \subset \Omega$ be an open set in $\mathbb{R}^3$ and S be a Lipschitz surface in $\mathbb{R}^3$ such that $\mathcal{C}(u) \cap \mathcal{O} \subset S$. Then the set $u(\mathcal{C}(u) \cap \mathcal{O})$ has Lebesgue measure zero.*

Proof of Lemma 17. Since S is a Lipschitz surface, up to work in local charts, we can assume that there is a homeomorphism $\Phi : B_3 \to \mathcal{O}$ which is Lipschitz with constant C_Φ and such that

$$\Phi\left(\mathcal{D}\right) = S,$$

where $\mathcal{D}$ is the disc defined by

$$\mathcal{D} := \{x \in B_3 \mid x_3 = 0\}.$$

Define the function $\hat{u} : \mathcal{D} \to \mathbb{R}$ by

$$\forall x \in \mathcal{D}, \quad \hat{u}(x) := u(\Phi(x)).$$

Then we have

$$\hat{u}\left(\Phi^{-1}(\mathcal{C}(u))\right) = u(\mathcal{C}(u)).$$

By (46), we obtain for every $x, y \in \Phi^{-1}(\mathcal{C}(u))$,

$$\begin{aligned} |\hat{u}(y) - \hat{u}(x)| &= |u\left(\Phi(y)\right) - u\left(\Phi(x)\right)| \\ &\leq \sigma \left|\Phi(y) - \Phi(x)\right|^2 \\ &\leq \sigma C_\Phi |y - x|^2. \end{aligned}$$

Lemma 2 yields the result. □

4.2 Image of the singular critical points

Denote by C the set of $x \in \Sigma(u)$ such that $0 \in D^+u(x)$. The following result holds.

Lemma 18. *The set $u(C)$ has Lebesgue measure zero.*

Proof of Lemma 18. From Theorem 6, we know that $\Sigma(u)$ is countably 2-rectifiable. This means that it is included in a countable union of Lipschitz surfaces. We conclude easily by Lemma 17. □

4.3 Image of the stationary critical set

Denote by A the set of points $x \in \Omega \setminus \Sigma(u)$ such that $\partial_L u(x) = \{0\}$ $(= \partial u(x))$ and $L_v(x, 0) = 0$. We must prove the following result.

Lemma 19. *The set $u(A)$ has Lebesgue measure zero.*

Proof of Lemma 19. We note that, for every $x \in A$, we have

$$H(x,0) = 0 \quad \text{and} \quad \frac{\partial H}{\partial p}(x,0) = 0. \tag{47}$$

Define the set $A_1 \subset A$ by

$$A_1 := \left\{ x \in A \mid \frac{\partial H}{\partial x}(x,0) \neq 0 \right\}.$$

If $x \in A_1$, then by the Implicit Function Theorem, there exists an open neighborhood $\mathcal{V} \in \Omega$ of x such that the set of $x \in \mathcal{V}$ verifying $H(x,0) = 0$ is a surface S of class at least C^4. But the set of $x \in A \cap \mathcal{V}$ is necessarily included in S. From Lemma 17, we deduce that the set $u(A \cap \mathcal{V})$ has measure zero. Then, $u(A_1)$ has measure zero too[5]. Denote by A_2 the complement of A_1 in A, that is,

$$A_2 := \left\{ x \in A \mid \frac{\partial H}{\partial x}(x,0) = 0 \right\}.$$

We must now prove that $u(A_2)$ has measure zero. The map $x \mapsto \frac{\partial H}{\partial x}(x,0)$ is at least C^3 from $\mathbb{R}^3$ into $\mathbb{R}^3$. Denote by f_1, f_2, f_3 its three coordinates. Define the set $A_3 \subset A_2$ by

$$A_3 := \left\{ x \in A_2 \mid \nabla f_1(x) \neq 0 \right\}.$$

If $x \in A_3$, then by the Implicit Function Theorem, there exists a neighborhood $\mathcal{V}$ of x such that the set of $x \in \mathcal{V}$ verifying $f_1(x) = 0$ is a surface S of class at least C^3. But the set of $x \in A_2 \cap \mathcal{V}$ is necessarily included in S. Thus, from Lemma 17, we deduce that $u(A_3 \cap \mathcal{V})$ has measure zero. Set

$$A_4 := \left\{ x \in A_2 \mid \nabla f_2(x) \neq 0 \right\},$$

and

$$A_5 := \left\{ x \in A_2 \mid \nabla f_3(x) \neq 0 \right\},$$

By the same reasoning as before, both sets $u(A_4 \cap \mathcal{V})$ and $u(A_5 \cap \mathcal{V})$ have measure zero. So, it remains to prove that the set $u(A_6)$ has measure zero, where A_6 is defined by

$$A_6 := \left\{ x \in A_2 \mid \frac{\partial^2 H}{\partial x^2}(x,0) = 0 \right\}.$$

Any point in A satisfies $\frac{\partial H}{\partial p}(x,0) = 0$. Hence, if we denote by g_1, g_2, g_3 the three coordinates of the map $x \mapsto \frac{\partial H}{\partial p}(x,0)$, we can define the sets $A_7, A_8, A_9 \subset A_6$ as follows:

$$A_j := \left\{ x \in A_6 \mid \nabla g_j(x) \neq 0 \right\}, \quad \text{for} \quad j = 7,8,9.$$

By the same reasoning as before, the sets $u(A_7), u(A_8), u(A_9)$ have measure zero. It remains to study the set $u(A_{10})$, where

$$A_{10} := \left\{ x \in A_6 \mid \frac{\partial^2 H}{\partial x \partial p}(x,0) = 0 \right\}.$$

By the same reasoning as before, we are leaded to consider the image of A_{11} constituted of $x \in A$ such that

$$H(x,0) = 0, \quad \frac{\partial H}{\partial x}(x,0) = 0, \quad \frac{\partial^2 H}{\partial x^2}(x,0) = 0, \quad \frac{\partial^2 H}{\partial x \partial p}(x,0) = 0,$$

[5] In fact, every point $x \in A_1$ possesses an open neighborhood $\mathcal{V}_x$ in Ω such that $u(A_1 \cap \mathcal{V}_x)$ has measure zero. Therefore, by local compactness of the open set $\cup_x \mathcal{V}_x$, there exists a countable family of points $\{x_i\}$ in A_1 such that

$$A_1 \subset \bigcup_i \mathcal{V}_{x_i}.$$

We deduce easily that $u(A_1)$ has measure zero.

and

$$\frac{\partial^3 H}{\partial x^2 \partial p}(x,0)=0, \quad \frac{\partial^4 H}{\partial x^4}(x,0)=0, \quad \frac{\partial^4 H}{\partial x^3 \partial p}(x,0)=0.$$

Recall that H is at least C^4 on the relatively open set Ω and satisfies the assumption (H3). Hence, by Taylor's formula, there are some constants $\alpha>0$, $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5 \geq 0$, and some continuous function $\epsilon : B_3 \times B_3 \rightarrow \mathbb{R}$ verifying

$$\lim_{(h,p)\rightarrow(0,0)} \epsilon(h,p)=0, \tag{48}$$

such that one has

$$H(x+h,p) \geq \alpha|p|^2 - \alpha_1|p|^3 - \alpha_2|h||p|^2 - \alpha_3|h|^2|p|^2 - \alpha_4|h||p|^3 - \alpha_5|p|^4 - \epsilon(h,p)|(h,p)|^4,$$

for every $x \in A_{11}$ and every $p,h \in B_3$. We deduce that there is $\mu>0$ such that

$$H(x+h,p) \geq |p|^2\left(\frac{\alpha}{2} - \alpha_2|h| - \alpha_3|h|^2\right) - \epsilon(h,p)|(h,p)|^4,$$

for every $x \in A_{11}$, every $h \in B_3$, and every $p \in B_3$ such that $|p| \leq \mu$. By (48), we deduce easily that there are $\rho, K>0$ such that for every $h,p \in B_3$ satisfying $|h|,|p| \leq \rho$ and every $x \in A_{11}$, we have

$$H(x+h,p) \leq 0 \Longrightarrow |p| \leq K|h|^2. \tag{49}$$

We are now ready to conclude. Since the mapping $x \mapsto \partial u(x)$ is upper semicontinuous on Ω, for every $x \in A_{11}$, there is an open ball $\mathcal{B}_x$ centered at x of radius less than ρ such that

$$|\zeta| \leq \rho, \quad \forall y \in \mathcal{B}_x, \quad \forall \zeta \in \partial u(y). \tag{50}$$

Let $y,z \in \mathcal{B}_x \cap A_{11}$ be fixed. By the Lebourg Mean Value Theorem (see Theorem 5), there is $t \in (0,1)$ and $\zeta \in \partial u\,(ty+(1-t)z)$ such that

$$u(z)-u(y)=\langle \zeta, z-y\rangle.$$

But since y belongs to A_{11} and $H\,(ty+(1-t)z) \leq 0$, (49) yields

$$|\zeta| \leq K(1-t)^2|z-y|^2.$$

Therefore, by Cauchy-Schwarz's inequality, we obtain

$$u(y)-u(z) \leq |\zeta||y-z| \leq K|y-z|^3$$

Lemma 3 in dimension 3 concludes the proof. □

4.4 Proofs of Theorems 2, 3 and 4

Denote by B the set of critical points of u which do not belong to $A \cup \Sigma(u)$, we have to prove that, under assumptions of Theorems 2, 3, 4, we have $\mathrm{meas}(u(B))=0$.

Proof of Theorem 2. By analyticity, either $H(x,0)=0$ for every $x \in M$, or the set of points $x \in M$ satisfying $H(x,0)=0$ is stratified by a locally finite family of analytic submanifolds of dimension less or equal than two (see [24] or [34]). In the second case, we can apply Lemma 17 for each submanifold and then conclude, while the first case is a corollary of Theorem 3 (under the assumption (H4)). □

Proof of Theorem 3. Assume that (H4) holds. By Hamiltonian-Lagrangian duality, this means that

$$\max_{v\in\mathbb{R}^3}\{-L(x,v)\} = H(x,0) = 0, \quad \forall x \in \Omega. \tag{51}$$

Hence, we have by convexity of L,

$$L(x,v) \geq 0, \quad \forall x \in \Omega, \quad \forall v \in \mathbb{R}^3, \tag{52}$$

and

$$L(x,v) = 0 \Longleftrightarrow \frac{\partial L}{\partial v}(x,v) = 0. \tag{53}$$

Thus, we have for every $v \in \mathbb{R}^n$,

$$L(x,v) = 0 \Longleftrightarrow \frac{\partial L}{\partial x}(x,v) = 0. \tag{54}$$

Fix a point $x \in B$ and denote by v_x the unique $v \in \mathbb{R}^n$ such that $H(x,0) = 0 = -L(x,v)$. Since $x \notin A$, we know that $v_x \neq 0$. Let Π_x be the affine subspace in $\mathbb{R}^3$ such that $x \in \Pi_x$ and $v_x \perp \Pi_x$. In fact, doing a linear change of coordinates if necessary, we can assume that $x = 0$, Π_x is the vector space generated by the first $n-1$ vectors $e_1, \cdots, e_{n-1}$ of the canonical basis of $\mathbb{R}^n$, and that $v_x = e_n$. Define the vector field X on Ω by

$$X_x(y) = \left(\frac{\partial L}{\partial v}\right)^{-1}(y,0), \quad \forall y \in \Omega^6$$

For every $y \in \Pi_x \cap \Omega$, denote by $\Gamma_y(t)$ the solution of

$$\dot{\Gamma}_y(t) = X_x(\Gamma_y(t)), \quad \forall t \in (-\infty, \infty),$$

such that $\Gamma_y(0) = y$. We note that, by construction, we have for every $y \in \Pi_x \cap \mathcal{B}$,

$$\frac{\partial L}{\partial v}(\Gamma_y(t), \dot{\Gamma}_y(t)) = 0.$$

Then (53) and (54) yield

$$\frac{\partial L}{\partial x}(\Gamma_y(t), \dot{\Gamma}_y(t)) = 0, \quad \forall t \in (-\infty, \infty).$$

Consequently, the pair $(\Gamma_y, \dot{\Gamma}_y)$ satisfies the Euler-Lagrange equation. In fact, the mapping

$$\Gamma : (t,y) \in \mathbb{R} \times \Pi_x \mapsto \Gamma_y(t)$$

is of class C^1 and satisfies (since $x = 0$, $\Pi_x = \mathrm{SPAN}\{e_1, \cdots, e_{n-1}\}$ and $v_x = e_n$)

$$D\Gamma(0,x) = I_n.$$

Therefore, by the Inverse Function Theorem, there is a cylinder $\mathcal{C}_x$ of the form $(-\epsilon, \epsilon) \times (B(x,\epsilon) \cap \Pi_x)$ (with $\epsilon > 0$) and a neighborhood $\mathcal{V}_x$ of x in Ω such that

$$\Gamma : \mathcal{C}_x \longrightarrow \mathcal{V}_x$$

[6] Recall that for every $y \in \Omega$, the mapping $\mathcal{L}_y : v \in \mathbb{R}^3 \mapsto \left(\frac{\partial L}{\partial v}\right)(y,v) \in \mathbb{R}^3$ is a diffeomorphism of class C^1 from $\mathbb{R}^3$ into $\mathbb{R}^3$. Here, the vector $\mathcal{L}_y^{-1}(0)$ is denoted by $\left(\frac{\partial L}{\partial v}\right)^{-1}(y,0)$.

is a diffeomorphism of class C^1.
Consider now a critical point z of u in B satisfying

$$\Gamma^{-1}(z) \in (0,\epsilon) \times (B(x,\epsilon) \cap \Pi_x),$$

that is such that

$$z = \Gamma_y(t) \quad \text{for some pair} \quad (t,y) \in \mathcal{C}_x.$$

Since $z \notin \Sigma(u)$, u is differentiable at z and $\partial u(z) = \partial_L u(z) = \{0\}$. Moreover, from Lemma 11, there is a curve $\gamma_z : (-\infty, 0] \to M$ satisfying $\gamma_z(0) = z$,

$$u(z) - u(\gamma_z(t)) = \int_t^0 L(\gamma_z(s), \dot{\gamma}_z(s))\, ds, \quad \forall t \in (-\infty, 0],$$

and such that

$$\mathcal{L}(z, \dot{\gamma}_z(0)) = (z, p).$$

In other terms, the curve γ_z is the solution to the Euler-Lagrange equation satisfying

$$\gamma_z(0) = z \quad \text{and} \quad \frac{\partial L}{\partial v}(z, \dot{\gamma}_z(0)) = 0.$$

By the Cauchy-Lipschitz Theorem applied to the Euler-Lagrange equation, we deduce that the curve γ_z satisfies

$$\gamma_z(s) = \Gamma_y(t-s), \quad \forall s \in (-\epsilon - t, \epsilon - t).$$

As a consequence, we obtain that for every $s \in (-\epsilon - t, \epsilon - t)$,

$$\begin{aligned} \nabla u(\gamma_z(s)) &= \frac{\partial L}{\partial v}(\gamma_z(s), \dot{\gamma}_z(s)) \\ &= \frac{\partial L}{\partial v}\Big(\Gamma_y(t-s), \dot{\Gamma}_y(t-s)\Big) = 0 \end{aligned}$$

In this way, we proved that for every critical point $z = \Gamma(t,y) \in \Gamma(\mathcal{C}_x)$, we have

$$u(z) = u(y) \quad \text{and} \quad y \text{ is a critical point of } u.$$

This means that

$$u(B \cap \mathcal{C}_x) = u(B \cap (B(x,\epsilon) \cap \Pi_x)).$$

But Π_x is a plane, hence Lemma 17 yields that the Lebesgue measure of $u(B \cap \mathcal{C}_x)$ equals zero. We conclude easily by local compactness of Ω.
Assume now that (H5) holds. In this case, the set B is empty. Hence, we conclude by Lemmas 18 and 19. □

Proof of Theorem 4. Let $f \in C^2(M,\mathbb{R})$ and $c \in \mathbb{R}$ be fixed, we set

$$H(x,p) := H_0(x,p) + f(x) - c, \quad \forall (x,p) \in T^*M,$$

and we define the C^2 function $\psi : M \to \mathbb{R}$ by

$$\psi(x) := H(x,0), \quad \forall x \in M.$$

We recall that a Morse function on M is a function in $C^2(M,\mathbb{R})$ whose critical points are all nondegenerate. Theorem 4 is based on the following result.

Lemma 20. *If ψ is a Morse function, then any viscosity solution of (2) satisfies the generalized Sard Theorem.*

Before proving the lemma, we recall that, if ψ is a Morse function, then each critical point of ψ is isolated in M. Thus the set $\mathcal{C}(\psi)$ is at most countable and its complement is an open subset of M.

Proof of Lemma 20. Let u be a viscosity solution of (2). We know, by Lemma 18, that $u\left(\mathcal{C}(u) \cap \Sigma(u)\right)$ has Lebesgue measure zero. The set $u(\mathcal{C}(\psi))$ has obviously measure zero. Therefore, it remains to prove that the set

$$u\left(\mathcal{C}(u) \cap (M \setminus \mathcal{C}(\psi)) \setminus \Sigma(u)\right)$$

has Lebesgue measure zero. But, if a point $x \in \mathcal{C}(u) \setminus \Sigma(u)$ is not a critical point of ψ, then the level set $\{\psi(y) = 0\}$ is locally a submanifold of M of dimension 2. Thus the set of y such that $0 \in \partial_L u(y)$ is locally contained in a surface of class C^2. Lemma 17 completes the proof. $\square$

Return to the proof of Theorem 4. In fact, we conclude easily by the fact that the set of Morse function is an open dense subset of $C^2(M, \mathbb{R})$ in the Whitney C^2 topology, see [22]. $\square$

5 A counterexample to Theorem 2 in dimension 4

The aim of this section is to provide a counterexample to Theorem 2 in the case of an analytic Hamiltonian in dimension four. Roughly speaking, the idea is to consider a solution to the Eikonal equation in the hyperbolic space U_4 (corresponding to the Poincaré half-space model for the hyperbolic space of dimension 4) which is the upper half-space in $\mathbb{R}^4$ defined in coordinates $x = (x_1, x_2, x_3, y)$ by $\{y > 0\}$ and equipped with the Riemannian metric

$$g = \frac{dx_1^2 + dx_2^2 + dx_3^3 + dy^2}{y^2}.$$

We recall that the geodesics in the upper half-space model of the hyperbolic space are given by the vertical half-lines and the semicircles with centers on the $y = 0$ hyperplane and whose the convex hull contains a vertical half-line (see [30]). Define the 1-form ω on U_4 by

$$\omega := dx_3,$$

and the real-analytic Hamiltonian $H : U_4 \times \left(\mathbb{R}^4\right)^* \to \mathbb{R}$ by

$$\forall (x,p) \in U_4 \times \left(\mathbb{R}^4\right), \quad H(x,p) := \|p + \omega(x)\|^2 - 1.$$

We leave the reader to verify that H satisfies the assumptions (H1)-(H3). We are going to prove that a viscosity solution $u : U_4 \to \mathbb{R}$ of the Hamilton-Jacobi equation

$$H(d_x u) = 0, \quad \forall x \in U_4. \tag{55}$$

provides a counterexample to Theorem 2 in dimension 4, that is, is such that its set of critical values $u(\mathcal{C}(u))$ contains an interval of positive length. More precisely, we will prove the following result:

Proposition 11. *The Hamilton-Jacobi equation (55) admits at least one viscosity solution which is at least $C^{1,1}_{loc}$ and such that the set $u(\mathcal{C}(u))$ contains an interval of positive length.*

Proof of Proposition 11. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a function of class C^2 whose the critical set $\mathcal{C}(f)$ is connected and such that its image $f(\mathcal{C}(f))$ is an interval with positive length (see for example [6] or [19]). Without loss of generality we can assume that $f = 0$ outside the ball centered at the points of coordinates $(0,0,1)$ with radius $1/2$ and that there is some constant $\epsilon \in (0,1)$ that will be chosen below as small as necessary) such that

$$|f(z)|, |\nabla f(z)| \leq \epsilon, \quad \forall z \in \mathbb{R}^3,$$

and the mapping $z \mapsto \nabla f(z)$ is globally Lipschitz with constant ϵ. Set

$$S := \{x = (x_1, x_2, x_3, y) \in U_4 \mid x_3 = f(x_1, x_2, y)\}$$

and

$$C := \{x \in U_4 \mid x_3 = f(x_1, x_2, y) \text{ and } (x_1, x_2, y) \in \mathcal{C}(f)\}.$$

First, we consider the eikonal equation

$$\begin{cases} \|d_x v\|^2 = 1, & \forall x \in U_4, \\ u(x) = 0, & \forall x \in S. \end{cases} \tag{56}$$

Define $v : U_4 \to \mathbb{R}$ by

$$v(x) := \Delta_S(x), \quad \forall x \in U_4,$$

where Δ_S denotes here the signed distance funtion to the set S with respect to the distance induced by g. Since U_4 has a negative (constant) curvature, the following result holds:

Lemma 21. *The function v is a viscosity solution of (56). Moreover, if $\epsilon > 0$ is small enough, v is $C^{1,1}_{loc}$ on U_4.*

Proof of Lemma 21. The fact that v is a viscosity solution of (56) is a basic fact in viscosity theory (see [33]). By classical results on the distance function in Riemannian geometry (see for instance [43]) together with Corollary 1, showing that v is $C^{1,1}_{loc}$ on U_4 is indeed equivalent to showing that for every pair of points $\bar{x} \neq \hat{x} \in S$, the two geodesics $\bar{\gamma}, \hat{\gamma} : \mathbb{R} \to U_4$ starting from $\bar{x}$ and $\hat{x}$ corresponding to the characteristics of the Hamilton-Equation (56) cannot intersect in positive or negative times. More precisely, denote by $\bar{V}$ (resp. $\hat{V}$) the unique vector in $\mathbb{R}^4$ which is orthogonal to $T_{\bar{x}}S$ (resp. $T_{\hat{x}}S$) and which points toward the same direction as the vector field $\frac{\partial}{\partial x_3}$. The geodesic $\bar{\gamma}$ (resp. $\hat{\gamma}$) is the one which satisfies $\bar{\gamma}(0) = \bar{x}$ and $\dot{\bar{\gamma}}(0) = \bar{V}$ (resp. $\hat{\gamma}(0) = \hat{x}$ and $\dot{\hat{\gamma}}(0) = \bar{V}$. We argue by contradiction. So, we assume that there are $\bar{x} \neq \hat{x}$ in S such that $\bar{\gamma}(t) = \hat{\gamma}(t)$ for some $t \in \mathbb{R}$. The geodesic $\bar{\gamma}$ (resp $\hat{\gamma}$) describes a semicircle $\bar{\mathcal{C}}$ (resp. $\hat{\mathcal{C}}$) which is tangent to $\bar{V}$ at $\bar{x}$ (resp. $\hat{V}$ at $\hat{x}$), centered on the $y = 0$ hyperplane, and whose the convex hull contains the vertical half-line. First, by construction, we can assume that the two points $\bar{x}, \hat{x}$ belong to the set

$$\tilde{S} := S \cap \left\{ (x \in U_4 \mid x_1^2 + x_2^2 + (y-1)^2 \leq \frac{1}{4} \right\}.$$

Moreover, up to do a change of coordinates and indeed for sake of simplicity, we can assume that $\bar{x}$ has the form $\bar{x} = (0,0,0,\bar{y})$ and $\bar{V} = (0,0,1,\bar{w})$ with $|\bar{w}| \leq \epsilon$. In that case the orbit of $\bar{\gamma}$ is given by the semicircle $\bar{\mathcal{C}}$ of equation

$$(x_3 - \bar{y}\bar{w})^2 + y^2 = \bar{y}^2 \left(1 + \bar{w}^2\right), \quad , x_1 = x_2 = 0, \quad y \geq 0.$$

If $\hat{x} = (\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{y})$ and $\hat{V} = (\hat{v}_1, \hat{v}_2, 1, \hat{w})$ then, by assumption on the function f, we have

$$|\hat{x}_3| \leq \epsilon \quad \text{and} \quad \left| \begin{array}{c} \hat{v}_1 \\ \hat{v}_2 \\ \hat{w} \end{array} \right| \leq \epsilon. \tag{57}$$

Moreover, since $\hat{x}_3 = f(\hat{x}_1, \hat{x}_2, \hat{y}) - f(0,0,\bar{y})$, we also have

$$|\hat{x}_3| \leq \epsilon \left| \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{y} - \bar{y} \end{pmatrix} \right| \tag{58}$$

and

$$|\hat{V} - \bar{V}| = \left| \begin{pmatrix} \hat{v}_1 \\ \hat{v}_2 \\ \hat{w} - \bar{w} \end{pmatrix} \right| \leq \epsilon \left| \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{y} - \bar{y} \end{pmatrix} \right|. \tag{59}$$

The center of the semicircle $\hat{C}$ is the point

$$\hat{C} = \begin{pmatrix} \hat{x}_1 - \lambda \hat{v}_1 \hat{w} \\ \hat{x}_2 - \lambda \hat{v}_2 \hat{w} \\ \hat{x}_3 - \lambda \hat{w} \\ 0 \end{pmatrix},$$

where

$$\lambda = \frac{-\hat{y}}{|\hat{V}|^2 - \hat{w}^2} = -\hat{y} + \delta \hat{y},$$

and δ is defined by

$$\delta := \left(\frac{\hat{v}_1^2 + \hat{v}_2^2}{1 + \hat{v}_1^2 + \hat{v}_2^2} \right) \hat{y}.$$

The radius of the circle $\hat{C}$ is given by

$$\hat{R}^2 = \frac{\hat{y}^2 |\hat{V}|^2}{|\hat{V}|^2 - \hat{w}^2}.$$

Hence the semicircle $\hat{\mathcal{C}}$ is included in the sphere $\hat{\mathcal{S}}$ of equation

$$(x_1 - \hat{x}_1 + \lambda \hat{v}_1 \hat{w})^2 + (x_2 - \hat{x}_2 + \lambda \hat{v}_2 \hat{w})^2 + (x_3 - \hat{x}_3 + \lambda \hat{w})^2 + y^2 = \hat{R}^2.$$

Let us distinguish two cases $\bar{y} \geq \hat{y}$ and $\bar{y} < \hat{y}$. In fact, we will treat only the first case, we second one is left to the reader.

First case: $\bar{y} \geq \hat{y}$
In this case, proving that the two semicircles $\bar{\mathcal{C}}$ and $\hat{\mathcal{C}}$ cannot intersect is equivalent to showing that the semicircle $\bar{\mathcal{C}}$ remains above the sphere $\hat{\mathcal{S}}$ in the upper half-space of $\mathbb{R}^4$. Hence, we need only to prove that

$$|\hat{C}M_1|^2, |\hat{C}M_2|^2 \geq \hat{R}^2,$$

where the points M_1, M_2 of $\bar{\mathcal{C}}$ are defined by

$$M_1 := \begin{pmatrix} 0 \\ 0 \\ \bar{y}\bar{\omega} + \bar{y}\sqrt{1 + \bar{\omega}^2} \\ 0 \end{pmatrix} \quad \text{and} \quad M_2 := \begin{pmatrix} 0 \\ 0 \\ \bar{y}\bar{\omega} - \bar{y}\sqrt{1 + \bar{\omega}^2} \\ 0 \end{pmatrix}.$$

Let us compute $|\hat{C}M_1|^2$. We have

$$|\hat{C}M_1|^2 = (\hat{x}_1 - \lambda \hat{v}_1 \hat{w})^2 + (\hat{x}_2 - \lambda \hat{v}_2 \hat{w})^2 + \left(\bar{y}\bar{\omega} + \bar{y}\sqrt{1 + \bar{\omega}^2} - \hat{x}_3 + \lambda \hat{w} \right)^2$$

By (59) together with the fact that

$$\begin{aligned} \bar{y}\bar{\omega} - \hat{y}\hat{\omega} &= \bar{y}\bar{\omega} - \bar{y}\hat{\omega} + \bar{y}\hat{\omega} - \hat{y}\hat{\omega} \\ &= \bar{\omega}(\bar{y} - \hat{y}) + \hat{y}(\hat{\omega} - \bar{\omega}) \\ &\geq -\epsilon|\bar{y} - \hat{y}| - \frac{3}{2}|\hat{\omega} - \bar{\omega}| \quad \text{(since } |\omega| \leq \epsilon \text{ and } \hat{y} \leq 3/2), \end{aligned}$$

and remenbering that $\epsilon < 1$, the square root of last term is greater than the quantity

$$\bar{y}\sqrt{1+\bar{\omega}^2} - \epsilon|\bar{y}-\hat{y}| - 3\left|\hat{V}-\bar{V}\right| - |\hat{x}_3|.$$

Thus, since $\bar{y}\sqrt{1+\bar{\omega}^2} \leq (3/2)\sqrt{1+\epsilon^2} < 3$, the last term is greater than

$$\bar{y}^2(1+\bar{\omega}^2) - 6\epsilon|\bar{y}-\hat{y}| - 18\left|\hat{V}-\bar{V}\right| - 6|\hat{x}_3|.$$

On the other hand, since $|\lambda| \leq \hat{y} \leq 3/2$ and $|\hat{w}| \leq \epsilon < 1$, the square roots of the first two terms are respectively greater than

$$|\hat{x}_1| - \frac{3}{2}\left|\hat{V}-\bar{V}\right| \quad \text{and} \quad |\hat{x}_2| - \frac{3}{2}\left|\hat{V}-\bar{V}\right|.$$

Thus, since $|\hat{x}_1|, |\hat{x}_2| \leq 1/2$, they are respectively greater than

$$\hat{x}_1^2 - \frac{3}{2}\left|\hat{V}-\bar{V}\right| \quad \text{and} \quad \hat{x}_2^2 - \frac{3}{2}\left|\hat{V}-\bar{V}\right|.$$

To summarize, we proved that

$$|CM_1|^2 \geq \bar{y}^2(1+\bar{\omega}^2) + \hat{x}_1^2 + \hat{x}_2^2 - 6\epsilon|\bar{y}-\hat{y}| - 21|\hat{V}-\bar{V}| - 6|\hat{x}_3|.$$

In consequence, in order to prove that $|CM_1|^2 \geq \hat{R}^2$, since

$$\hat{R}^2 = \frac{\hat{y}^2|\hat{V}|^2}{|\hat{V}|^2 - \hat{w}^2} = \hat{y}^2 + \hat{y}^2\frac{\hat{\omega}^2}{1+\hat{v}_1^2+\hat{v}_2^2} \geq \hat{y}^2 + \hat{y}^2\hat{\omega}^2,$$

it is sufficient to show that

$$\Delta := \bar{y}^2 - \hat{y}^2 + \bar{y}^2\bar{\omega}^2 - \hat{y}^2\hat{\omega}^2 + \hat{x}_1^2 + \hat{x}_2^2 - 6\epsilon|\bar{y}-\hat{y}| - 21|\hat{V}-\bar{V}| - 6|\hat{x}_3| \geq 0.$$

But we notice that since $\bar{y} \geq \hat{y} \geq 1/2$ and $|\overline{w}|, |\hat{\omega}| \leq \epsilon < 1$, we have

$$\bar{y}^2 - \hat{y}^2 = (\bar{y}+\hat{y})(\bar{y}-\hat{y}) \geq \bar{y} - \hat{y},$$

and

$$\bar{y}^2\bar{\omega}^2 - \hat{y}^2\hat{\omega}^2 \geq \hat{y}^2(\bar{\omega}+\hat{\omega})(\bar{\omega}-\hat{\omega}) \geq -\frac{1}{2}|\bar{\omega}-\hat{\omega}| \geq -|\hat{V}-\bar{V}|.$$

Then we deduce that

$$\Delta \geq (\bar{y}-\hat{y}) + \hat{x}_1^2 + \hat{x}_2^2 - 6\epsilon|\bar{y}-\hat{y}| - 22|\hat{V}-\bar{V}| - 6|\hat{x}_3|.$$

Remenbering (58) and (59), it is clear that if ϵ is taken small enough, then Δ is greater than 0. The reasoning to show that $|\hat{C}M_2|^2 \geq \hat{R}^2$ for ϵ small enough being the same, it is left to the reader. □

Returning to the proof of Proposition 11, we define $u : U_4 \to \mathbb{R}$ by

$$\forall x \in U_4, \quad u(x) = u(x_1, x_2, x_3, y) := v(x) - x_3.$$

By construction, we verify that we have

$$H(d_x u) = H(d_x v - \omega(x)) - 1 = \|d_x v\|^2 - 1 = 0, \quad \forall x \in U_4.$$

Furthermore, we also have

$$d_x u = d_x v - \omega(x) = 0, \quad \forall x \in C.$$

Which means that C is included in the critical set $\mathcal{C}(u)$. But we have for every $x \in C$,

$$u(x) = u(x_1, x_2, f(x_1, x_2, y), y) = v(x) - x_3 = -x_3.$$

Hence we have

$$I := \{-f(x_1, x_2, y) \mid (x_1, x_2, y) \in \mathcal{C}(f)\} \subset u(\mathcal{C}(u)).$$

But the set $-I$ corresponds exactly to the set of critical values of f. Therefore, since $f : \mathbb{R}^3 \to \mathbb{R}$ is a C^2 counterexample to the Sard Theorem, the set I is an interval of positive length (notice that it is an interval since we assumed that $\mathcal{C}(f)$ is connected). This completes the proof of Proposition 11. □

We proved Lemma 21 in a very naive way. Another possibility to prove Lemma 21 would be to use the fact that U_4 is a Riemannian manifold of negative constant curvature (see [8]).

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Appendix A

Let $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be an Hamiltonian of class C^k (with $k \geq 2$) which satisfies (H1)-(H3) together with the following assumption:

(H4) For every $x \in \mathbb{R}^n, H(x, 0) < 0$.

Let Ω be a open set in $\mathbb{R}^n$ with compact boundary, denoted by $\partial\Omega$, of class C^l (with $l \geq 2$). We are interested in the following Dirichlet type Hamilton-Jacobi equation

$$\begin{cases} H(x, du(x)) = 0, & \forall x \in \Omega, \\ u(x) = 0, & \forall x \in \partial\Omega. \end{cases} \tag{60}$$

The Lagrangian $L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ associated to H is defined by,

$$\forall (x, v) \in \mathbb{R}^n \times \mathbb{R}^n, \quad L(x, v) := \max_{p \in \mathbb{R}^n} \{\langle p, v\rangle - H(x, p)\}.$$

It is of class C^k (see [10, Corollary A.2.7 p. 287]) and satisfies the properties (L1)-(L4) given in Section 2.1. For any $x, y \in \overline{\Omega}$, we set

$$l(x, y) := \inf \left\{ \int_0^T L(\gamma(t), \dot{\gamma}(t))dt \mid T \geq 0, \gamma \in \Omega_T(x, y) \right\},$$

where for every $T \geq 0$, $\Omega_T(x, y)$ denotes the set of locally Lipschitz curves $\gamma : [0, T] \to \overline{\Omega}$ such that $\gamma(0) = x$ and $\gamma(T) = y$. The result that we intend to prove here is the following:

Theorem 9. *The function $u : \overline{\Omega} \to \mathbb{R}$ given by*

$$u(x) := \inf \{l(y, x) \mid y \in \partial\Omega\}, \quad \forall x \in \overline{\Omega}, \tag{61}$$

is well-defined and continuous on $\overline{\Omega}$. Moreover, it is the unique viscosity solution of (60), and if $k \geq 2n^2 + 4n + 1$, the set $u(\mathcal{C}(u))$ has Lebesgue measure zero.

The proof of Theorem 9 is based on the following immediate corollary of [39, Theorem 3] (with a lower bound on k taken from [14]).

Theorem 10. *Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$, $\mathcal{Q}$ be an open subset of $\mathbb{R}^N$, and $\mathcal{C}$ be a compact subset of $\mathcal{Q}$. Let $\varphi : \mathcal{O} \times \mathcal{Q} \to \mathbb{R}$ be a smooth function of class C^k and $f : \mathcal{O} \to \mathbb{R}$ the function defined by*

$$f(x) := \min_{q \in \mathcal{Q}} \{\varphi(x, q)\}, \quad \forall x \in \mathcal{O}.$$

Assume that for every $x \in \mathcal{Q}$, there is $c \in \mathcal{C}$ such that $f(x) = \varphi(x, c)$ and that $k \geq 2n+N(n+1)$. Then $f(\mathcal{C}(f))$ has Lebesgue measure zero.

It has to be noticed that the lower bound on the regularity of φ comes from the fact that Theorem 10 is a corollary of the classical Sard Theorem applied to a mapping from a manifold of dimension $2n + N(n + 1)$ into $\mathbb{R}$ (see [14]). The proof of Theorem 9 that we will sketch below is adapted from a new proof given by Albert Fathi of the Sard Theorem for the distance function on Riemannian manifolds (see [14]). Let us just sketch the proof of Theorem 9.

The fact that u is well-defined and continuous is easy and left to the reader. The fact that, under the additional assumption (H4), the function u given by (61) is a viscosity solution of (60) is a standard result in viscosity theory (see [33, Theorem 5.4 p. 134]). The fact that, thanks to (H4), u is indeed the unique viscosity solution is less classical; we refer the reader to [27] or [5] for its proof. Let us now collect, without proof, some other properties satisfied by the function u:

(i) The function u is locally semiconcave on Ω.

(ii) For every $x \in \Omega$ and every $p \in \partial_L u(x)$, there are $T_{x,p} > 0$ and a curve $\gamma_{x,p} : [-T_{x,p}, 0] \to \mathbb{R}$ such that

$$\gamma_{x,p}(0) = x, \quad \frac{\partial L}{\partial v}(\gamma_{x,p}(0), \dot{\gamma}_{x,p}(0)) = p$$

and

$$u(x) - u(\gamma(t)) = \int_t^0 L\left(\gamma_{x,p}(s), \dot{\gamma}_{x,p}(s)\right) ds, \quad \forall t \in [-T_{x,p}, 0].$$

Moreover, the time $T_{x,p}$ can be chosen in such a way that

$$\gamma_{x,p}(-T_{x,p}) \in \partial\Omega.$$

In that case, the vector $\dot{\gamma}_{x,p}(-T_{x,p})$ is necessarily transverse to the submanifold $\partial\Omega$ (while the vector $\frac{\partial L}{\partial v}\left(\gamma_{x,p}(-T_{x,p}), \dot{\gamma}_{x,p}(-T_{x,p})\right)$ is orthogonal to $\partial\Omega$).

(iii) Since u is a viscosity solution of (60), we have for every $T > 0$ and every locally Lipschitz curve $\gamma : [-T, 0] \to \overline{\Omega}$ satisfying $\gamma(0) = x$,

$$u(x) - u(\gamma(-T)) \leq \int_{-T}^0 L\left(\gamma(s), \dot{\gamma}(s)\right) ds.$$

(iv) As a consequence, we have for every $x \in \Omega$, every $p \in D^- u(x)$, every $T > 0$, and every locally Lipschitz curve $\gamma : [-T, 0] \to \overline{\Omega}$ satisfying $\gamma(0) = x$ and $\gamma(-T) \in \partial\Omega$,

$$\int_{-T_{x,p}}^0 L\left(\gamma_{x,p}(t), \dot{\gamma}_{x,p}(t)\right) dt \leq \int_{-T}^0 L\left(\gamma(s), \dot{\gamma}(s)\right) ds.$$

(v) In addition, for every $x \in \Omega$ and every $p \in \partial_L u(x)$, the curve $\gamma_{x,p}$ is the solution of the Euler-Lagrange equation satisfying

$$\gamma_{x,p}(0) = x \quad \text{and} \quad \frac{\partial L}{\partial v}(\gamma_{x,p}(0), \dot{\gamma}_{x,p}(0)) = p.$$

Furthermore, we mention that, since H is of class C^k, the Euler-Lagrange flow is of class C^{k-1} (see [16]). Set for every $(x,p) \in \Omega \times \mathbb{R}^n$,

$$T(x,p) := \inf \left\{ T \geq 0 \mid \phi^L_{-T}\left(\mathcal{L}^{-1}(x,p)\right) \in \partial\Omega \right\}.$$

(we set $T(x,p) = \infty$ if the set of $t \geq 0$ such that $\phi^L_{-T}\left(\mathcal{L}^{-1}(x,p)\right)$ is empty).

Fix $\bar{x} \in \Omega$ and $\bar{p} \in \partial_L u(\bar{x})$. Since the vector $\dot{\gamma}(-T_{\bar{x},\bar{p}})$ is transverse to $\partial\Omega$, there is a neighborhood of $\bar{p}$ in $\mathbb{R}^n$ where the mapping $p \mapsto T(\bar{x},p)$ is of class C^{k-1}. In fact, since the set $\partial_L u(\bar{x})$ is compact and since the multivalued mapping $x \mapsto \partial_L u(x)$ is upper semicontinuous, there is $\delta > 0$ such that the following properties are satisfied:

(P1) $\forall x \in B(\bar{x},\delta), \quad \partial_L u(x) \subset \partial_L u(\bar{x}) + \bar{B}_{\frac{\delta}{2}}$.

(P2) The mapping $(x,p) \mapsto T(x,p)$ is of class C^{k-1} on the open set $\mathcal{Q} := B(\bar{x},\delta) \times (\partial_L u(\bar{x}) + B_\delta)$.

We notice that, by construction, we have for every $x \in B(\bar{x},\delta)$,

$$u(x) = \min \left\{ \int_{-T(x,q)}^{0} L\left(\gamma_{y,v}(t), \dot{\gamma}_{y,v}(t)\right) dt \mid q \in \mathcal{Q} \right\}.$$

Moreover, we know that for every $x \in B(\bar{x},\delta)$, the minimum in the formula above is indeed attained in the compact set

$$\mathcal{C} := \partial_L u(\bar{x}) + \bar{B}_{\frac{\delta}{2}}.$$

Theorem 10 yields the result.

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